FACULTY OF MECHATRONICS,
INFORMATICS AND INTERDISCIPLINARY
STUDIES TUL

Poroelasticity: Theory, numerical solution and applications

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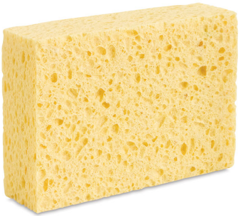
Outline of presentation

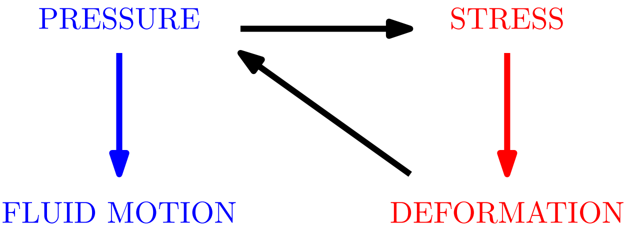
1. Mathematical modelling of poroelasticity
2. Well-posedness
3. Approximation
4. Iterative splitting
5. Applications in rock hydro-mechanics

Mathematical modelling

What is poroelasticity I

poroelasticity = porous media flow + elasticity

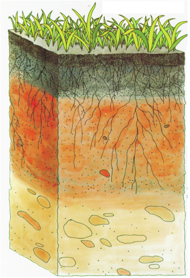


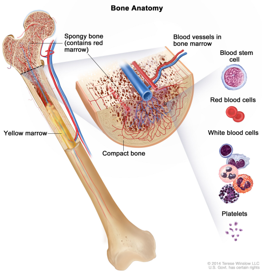


What is poroelasticity II

Examples of deformable porous media

- soils, rocks
- biological tissues
- building materials

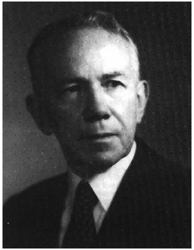




Overview of literature

- K. Terzaghi (1923, 1925)
- M. A. Biot (1941)
- F. Gassmann (1951)
- E. Detournay and A. H.-D. Cheng (1993)
- A. Verruijt (2016)

K. Terzaghi
(1883-1963)

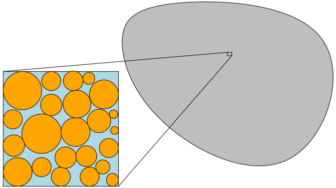


M. A. Biot
(1905-1985)

Derivation of Biot model

General assumptions

- macroscopically homogeneous material
- 2 components: solid skeleton + fluid filled pores
- linearly elastic solid skeleton
- interconnected pores

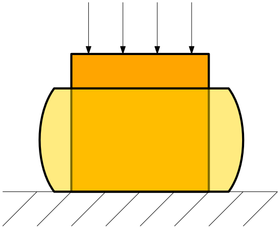


porosity:

$$\phi = \frac{V_f}{V} = 1 - \frac{V_s}{V}$$

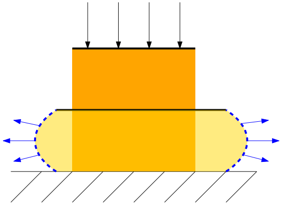
Derivation of Biot model

Drained and undrained deformations



Undrained deformation:

- fluid is entrapped in the domain
- stress is carried by
    - compression of fluid
    - compression of solid particles
    - rearrangement of solid skeleton



Drained deformation:

- fluid can flow in/out
- stress is carried only by solid particles and skeleton
- consolidation takes time due to low permeability

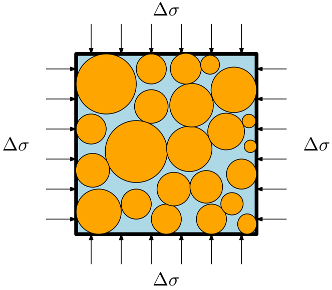
Derivation of Biot model

Undrained compression test

- hydraulically isolated sample
- applied stress $\Delta\sigma = \underbrace{\Delta p}_{\text{pore pressure}} + \underbrace{(\Delta\sigma - \Delta p)}_{\text{effective stress}}$
- volume change of porous medium:

$$\frac{\Delta V}{V} = \underbrace{-C_s \Delta p}_{\substack{\text{deformation due}\\\text{to pore pressure}}} \underbrace{-C_m(\Delta\sigma - \Delta p)}_{\substack{\text{deformation (rearrange-}\\\text{ment of particles) at con-}\\\text{stant pore pressure}}}$$

$$= \underbrace{-\phi C_f \Delta p}_{\text{deformation of fluid}} \underbrace{-(1-\phi)C_s\left(\Delta p + \frac{\Delta\sigma - \Delta p}{1-\phi}\right)}_{\text{deformation of solid particles}}$$



compressibilities:
C_f - fluid
C_s - solid particles
C_m - porous medium

Derivation of Biot model

Undrained compression test

Skempton pore pressure coefficient:

$$B := \frac{\Delta p}{\Delta \sigma} = \frac{C_m - C_s}{(C_m - C_s) + \phi(C_f - C_s)} < 1 \qquad \text{if } C_f, C_s > 0$$

Decomposition of total stress tensor into effective stress and pore pressure:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - \alpha p \boldsymbol{I}, \qquad \alpha = \text{Biot-Willis coefficient}$$

Sign convention: compresive stress negative
For isotropic media:

$$\frac{\Delta V}{V} = \varepsilon = C_m \Delta \sigma' = C_m(\Delta \sigma + \alpha \Delta p) \quad \Rightarrow \quad \alpha = 1 - \frac{C_s}{C_m}$$

Derivation of Biot model

Balance laws

Assumption: Time scale of mechanics $\ll$ time scale of porous media flow
Equilibrium equations for total stress:

$$-\operatorname{div}\boldsymbol{\sigma} = \boxed{-\operatorname{div}\boldsymbol{\sigma}' + \alpha\nabla p = \boldsymbol{f}}$$

Hooke's law:

$$\boldsymbol{\sigma}' = \boldsymbol{C}\varepsilon(\boldsymbol{u}) \quad \Leftrightarrow \quad \varepsilon(\boldsymbol{u}) = \boldsymbol{C}^{-1}\boldsymbol{\sigma}'$$

Isotropic media:

$$\boldsymbol{C}\varepsilon = 2G\varepsilon + (K - \tfrac{2}{3}G)(\operatorname{tr}\varepsilon)\boldsymbol{I}, \qquad K = \frac{1}{C_m}$$

K...bulk modulus
G...shear modulus

Derivation of Biot model

Balance laws

Conservation of mass:

$$\partial_t (\phi\rho_f) + \operatorname{div}(\phi\rho_f \boldsymbol{v}) = 0 \qquad \partial_t ((1-\phi)\rho_s) + \operatorname{div}((1-\phi)\rho_s \boldsymbol{w}) = 0$$

Density of fluid and solid:

$$\rho_f(p) = \rho_{f0} \exp(C_f p) \qquad \rho_s(p,\sigma) = \rho_{s0} \exp\left(\frac{C_s}{1-\phi}(\phi p - \sigma)\right)$$

Conservation of mass for porous medium:

$$\operatorname{div} \boldsymbol{w} + \operatorname{div}(\phi(\boldsymbol{v} - \boldsymbol{w})) + \phi(C_f - C_s)\partial_t p - C_s \partial_t \sigma = 0$$

$\boldsymbol{q} := \phi(\boldsymbol{v} - \boldsymbol{w})$ (specific discharge)

$\sigma := \sigma' - \alpha p = \operatorname{div} \boldsymbol{u}/C_m - \alpha p$ (isotropic stress)

$S := \phi C_f + (\alpha - \phi) C_s$ (storativity)

Derivation of Biot model

Balance laws

Conservation of mass of porous medium (storage equation):

$$\boxed{\alpha\partial_t \operatorname{div}\boldsymbol{u} + S\partial_t p = -\operatorname{div}\boldsymbol{q}}$$

Darcy's law:

$$\boldsymbol{q} = -\frac{\kappa}{\mu}\nabla(p - \underbrace{\rho_f g e}_{\text{hydrostatic pressure}}) = -k\nabla\underbrace{\left(\frac{p}{\rho_f g} - e\right)}_{\text{piezometric head}}$$

κ...permeability (generally tensor)
μ...fluid viscosity
e...elevation
k...hydraulic conductivity (generally tensor)
g...gravitational acceleration

Biot system

Assuming $\rho_f g = 1$, the poroelasticity equations read:

$$\left.\begin{aligned} -\operatorname{div}(\boldsymbol{C}\varepsilon(\boldsymbol{u})) + \alpha\nabla p &= \boldsymbol{f} \\ \partial_t\left(Sp + \alpha\operatorname{div}\boldsymbol{u}\right) - \operatorname{div}(k\nabla p) &= b \end{aligned}\right\} \text{ in } (0,T)\times\Omega$$

- Coupled linear elliptic-parabolic system with unknowns $(\boldsymbol{u}, p)$
- Initial condition - only for flow:

$$p(0,\cdot) = p_0 \text{ in } \Omega$$

- Boundary conditions:

$$\begin{aligned} \text{Flow:} \quad & p = p_D \text{ on } (0,T)\times\Gamma_{Df} \quad \boldsymbol{q}\cdot\boldsymbol{n} = q_N \text{ on } (0,T)\times\Gamma_{Nf} \\ \text{Mechanics:} \quad & \boldsymbol{u} = \boldsymbol{u}_D \text{ on } (0,T)\times\Gamma_{Dm} \quad \sigma'\boldsymbol{n} = \boldsymbol{t}_N \text{ on } (0,T)\times\Gamma_{Nm} \end{aligned}$$

Generalizations and departures from poroelasticity

- Dynamic elasticity...hyperbolic-parabolic system:

$$\underbrace{\rho \partial_{tt}^2 \boldsymbol{u}}_{\text{inertial term}} + \underbrace{\lambda^* \nabla \partial_t \operatorname{div} \boldsymbol{u}}_{\text{secondary consolidation term}} - \operatorname{div}(\boldsymbol{C} \varepsilon(\boldsymbol{u})) + \alpha \nabla p = \boldsymbol{f}$$

- Initial (reference) stress - important for nonlinear problems:

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}' - \boldsymbol{\sigma}_0) - \alpha p \boldsymbol{I}$$

- Hydro-mechanical parameter coupling, e.g. $k = k(\boldsymbol{u}, \boldsymbol{\sigma})$
- Unsaturated flow
- Non-linear mechanics: plasticity, damage, fracture mechanics
- Thermo-poro-elasticity
- ...

Well-posedness

Biot system

A priori estimate

- For simplicity we assume homogeneous b.c.: $p = 0, \ \boldsymbol{u} = \mathbf{0} \quad \text{on } (0,T) \times \partial\Omega$
- Differentiate elasticity equation w.r.t. to time, multiply by $\partial_t \boldsymbol{u}$, and integrate:

$$\int_\Omega \partial_t \boldsymbol{u} \partial_t \left(-\operatorname{div}(\boldsymbol{C}\varepsilon(\boldsymbol{u})) + \alpha \nabla p\right) =$$

$$= \int_\Omega \boldsymbol{C}\varepsilon(\partial_t \boldsymbol{u}) : \varepsilon(\partial_t \boldsymbol{u}) - \alpha \int_\Omega \partial_t p \operatorname{div} \partial_t \boldsymbol{u} = \int_\Omega \partial_t \boldsymbol{u} \cdot \partial_t \boldsymbol{f}$$

- Multiply storage (flow) equation by $\partial_t p$ and integrate:

$$\int_\Omega \partial_t p (\partial_t (Sp + \alpha \operatorname{div} \boldsymbol{u}) - \operatorname{div}(k \nabla p)) =$$

$$= S \int_\Omega |\partial_t p|^2 + \alpha \int_\Omega \partial_t p \operatorname{div} \partial_t \boldsymbol{u} + k \int_\Omega \partial_t \nabla p \cdot \nabla p = \int_\Omega b \partial_t p$$

- Sum both equations...

Biot system

A priori estimate

- Sum of both equations:

$$\int_\Omega \boldsymbol{C}\varepsilon(\partial_t \boldsymbol{u}) : \varepsilon(\partial_t \boldsymbol{u}) + S\int_\Omega |\partial_t p|^2 + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla p|^2 = \int_\Omega \partial_t \boldsymbol{u}\cdot\partial_t \boldsymbol{f} + \int_\Omega b\partial_t p$$

- Integrate w.r. to time:

$$\int_0^\tau\int_\Omega \boldsymbol{C}\varepsilon(\partial_t \boldsymbol{u}) : \varepsilon(\partial_t \boldsymbol{u}) + S\int_0^\tau\int_\Omega |\partial_t p|^2 + \frac{k}{2}\int_\Omega |\nabla p|^2(\tau)$$
$$= \int_0^\tau\int_\Omega (\partial_t \boldsymbol{u}\cdot\partial_t \boldsymbol{f} + b\partial_t p) + \frac{k}{2}\int_\Omega |\nabla p_0|^2$$

- If $\boldsymbol{C}\varepsilon : \varepsilon \geqslant 2G|\varepsilon|^2$, $S, k > 0$ then there is a constant $C = C(G, S, k) > 0$:

$$\int_0^T \left(\|\nabla\partial_t \boldsymbol{u}\|_2^2 + \|\partial_t p\|_2^2\right) + \sup_{\tau\in(0,T)} \|\nabla p\|_2^2(\tau) \leqslant C\int_0^T \left(\|\partial_t \boldsymbol{f}\|_2^2 + \|b\|_2^2\right) + \|\nabla p_0\|_2^2$$

Biot system

Weak solution

Weak formulation of Biot problem (B)

Find $\boldsymbol{u} \in H^1(0,T;\boldsymbol{H}_0^1(\Omega))$, $p \in L^\infty(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ s.t.

- $p(0,\cdot) = p_0$ in Ω;
- $\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ and a.e. $t \in (0,T)$:

$$\int_\Omega [\boldsymbol{C}\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{v}) - \alpha p(t)\,\mathrm{div}\,\boldsymbol{v}] = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v};$$

- $\forall q \in H_0^1(\Omega)$ and a.e. $t \in (0,T)$:

$$\int_\Omega [\partial_t\,(Sp(t) + \alpha\,\mathrm{div}\,\boldsymbol{u}(t))q + k\nabla p(t)\cdot\nabla q] = \int_\Omega b(t)q.$$

Biot system

Existence of weak solutions

Assumptions:

- $\forall \varepsilon : \ \boldsymbol{C}\varepsilon : \varepsilon \geqslant 2G|\varepsilon|^2 + (K - \frac{2}{3}G)|\operatorname{tr}\varepsilon|^2$
- $S, k > 0$
- $p_0 \in H_0^1(\Omega)$, $\boldsymbol{f} \in H^1(0,T;\boldsymbol{L}^2(\Omega))$, $b \in L^2(0,T;L^2(\Omega))$

Theorem

Under the above assumptions, problem (B) has a unique solution.

References:

- A. Ženíšek (1984): weak solutions, $S = 0$
- R. E. Schowalter (2000): strong and weak solutions, quasistatic/dynamic case

Approximation

Approximation of two-field formulation of Biot system I

Two-field (primal) formulation $(\boldsymbol{u}, p)$:

$$
\begin{aligned}
-\operatorname{div}(\boldsymbol{C}\varepsilon(\boldsymbol{u})) + \alpha\nabla p &= \boldsymbol{f} \\
\partial_t\left(Sp + \alpha \operatorname{div}\boldsymbol{u}\right) - \operatorname{div}(k\nabla p) &= b
\end{aligned}
$$

Temporal semidiscretization (e.g. implicit Euler's method with equidistant timestepping, $\Delta t = T/N$, $t_i = i\Delta t$, $i = 1, \ldots, N$):

$$
\begin{aligned}
(\boldsymbol{C}\varepsilon(\boldsymbol{u}(t_i)), \varepsilon(\boldsymbol{v})) - (\alpha p(t_i), \operatorname{div}\boldsymbol{v}) &= (\boldsymbol{f}(t_i), \boldsymbol{v}) \\
(\alpha \operatorname{div}\boldsymbol{u}(t_i), q) + (Sp(t_i), q) + \Delta t(k\nabla p(t_i), \nabla q) &= (\Delta t b(t_i) + Sp(t_{i-1}) + \alpha \operatorname{div}\boldsymbol{u}(t_{i-1}), q)
\end{aligned}
$$

In operator form:

$$
\begin{bmatrix} \mathcal{A} & -\mathcal{B}^\top \\ \mathcal{B} & \mathcal{C} \end{bmatrix}
\begin{bmatrix} \boldsymbol{u}(t_i) \\ p(t_i) \end{bmatrix}
=
\begin{bmatrix} \boldsymbol{f}_i \\ b_i \end{bmatrix}
$$

Approximation of two-field formulation of Biot system II

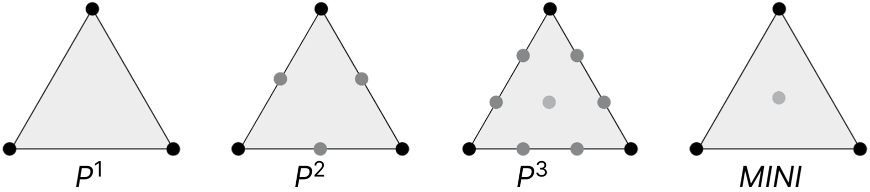
- Skew-symmetric saddle-point structure;
- For small S and $\Delta t\, k$, $\mathcal{C} \approx 0$... Biot $\approx$ Stokes:

$$\begin{bmatrix} \mathcal{A} & -\mathcal{B}^\top \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}(t_i) \\ p(t_i) \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i \\ b_i \end{bmatrix}$$

Finite element discretization:

- for sufficiently large S, arbitrary P^k/P^l finite elements work $(k, l = 1, 2, \ldots)$;
- for small S, spurious pressure oscillations can appear $\Rightarrow$ use Stokes-stable pair (e.g. Taylor-Hood P^{k+1}/P^k or MINI element)

Approximation of two-field formulation of Biot system III



- in geosciences, usually lowest order approximations are used
- stress and flux are computed using solution gradient
  $\Rightarrow$ worse FE approximation
- remedy: mixed/dual formulations

Approximation of three-field formulation of Biot system I

Three-field (primal-dual) formulation $(\boldsymbol{u}, p, \boldsymbol{q})$:

- additional unknown $\boldsymbol{q}$ given by Darcy's law

$$
\begin{aligned}
-\operatorname{div}(\boldsymbol{C}\varepsilon(\boldsymbol{u})) + \alpha \nabla p &= \boldsymbol{f} \\
\partial_t \left(Sp + \alpha \operatorname{div} \boldsymbol{u}\right) + \operatorname{div} \boldsymbol{q} &= b \\
k^{-1}\boldsymbol{q} + \nabla p &= \boldsymbol{0}
\end{aligned}
$$

Operator form of time-semidiscretized problem:

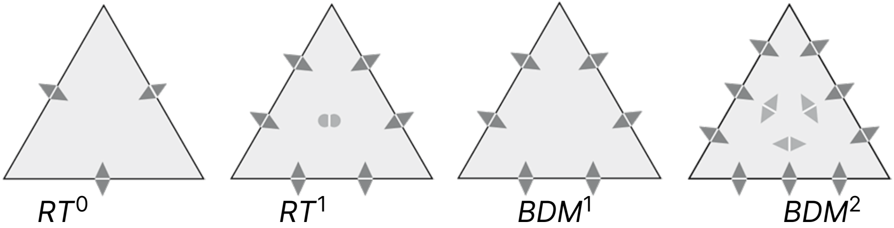
$$
\begin{bmatrix} \mathcal{A} & -\mathcal{B}^\top & 0 \\ \mathcal{B} & \mathcal{C} & \mathcal{D} \\ 0 & -\mathcal{D}^\top & \mathcal{E} \end{bmatrix}
\begin{bmatrix} \boldsymbol{u}(t_i) \\ p(t_i) \\ \boldsymbol{q}(t_i) \end{bmatrix}
=
\begin{bmatrix} \boldsymbol{f}_i \\ b_i \\ \boldsymbol{0} \end{bmatrix}
$$

- two-fold skew-symmetric saddle-point structure

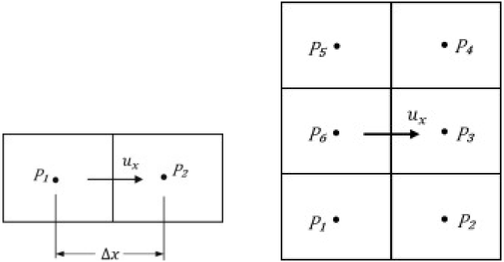
Approximation of three-field formulation of Biot system II

FE/FV discretization:

- stable pressure-flux pair: e.g. P^k/RT^k, P^k/BDM^{k+1}, $k = 0, 1, ...$, or finite volume methods P^0/TPFA, P^0/MPFA
- stable displacement-pressure pair: for lowest order pressure space, P^2/P^0 works in 2D, in 3D more delicate issue



Approximation of three-field formulation of Biot system III



Two-point / multi-point flux approximation FV schemes

Approximation of five-field formulation of Biot system I

Five-field (dual-dual) formulation $(\sigma, \boldsymbol{r}, \boldsymbol{u}, p, \boldsymbol{q})$:

- additional unknown σ given by Hooke's law
- symmetry of σ enforced weakly...Lagrange multiplier $\boldsymbol{r}$

$$
\begin{aligned}
-\operatorname{div} \sigma &= \boldsymbol{f} \\
\partial_t \left(Sp + \alpha \operatorname{tr}(\boldsymbol{C}^{-1}(\sigma + \alpha p \boldsymbol{I})) \right) + \operatorname{div} \boldsymbol{q} &= b \\
k^{-1} \boldsymbol{q} + \nabla p &= \boldsymbol{0} \\
\boldsymbol{C}^{-1}(\sigma + \alpha p \boldsymbol{I}) - \nabla \boldsymbol{u} + \operatorname{as}^* \boldsymbol{r} &= \boldsymbol{0} \\
\operatorname{as} \sigma &= \boldsymbol{0}
\end{aligned}
$$

Approximation of five-field formulation of Biot system II

Operator form of time-semidiscretized problem:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}^\top & \mathcal{C}^\top & \mathcal{D}^\top & 0 \\ -\mathcal{B} & 0 & 0 & 0 & 0 \\ -\mathcal{C} & 0 & 0 & 0 & 0 \\ \mathcal{D} & 0 & 0 & \mathcal{E} & \mathcal{F}^\top \\ 0 & 0 & 0 & -\mathcal{F} & \mathcal{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}(t_i) \\ \boldsymbol{r}(t_i) \\ \boldsymbol{u}(t_i) \\ p(t_i) \\ \boldsymbol{q}(t_i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{f}_i \\ b_i \\ \boldsymbol{0} \end{bmatrix}$$

- block-symmetric saddle-point problem

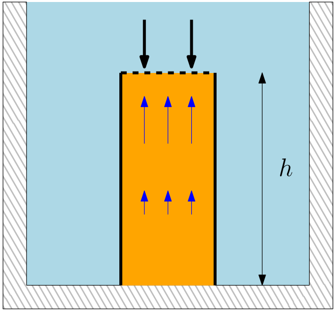
FE/FV discretization:

- stable mixed elasticity spaces: e.g. $\boldsymbol{\sigma} \in BDM^1$, $\boldsymbol{u} \in P^0$, $\boldsymbol{r} \in P^0$ or MPSA/P^0/P^0
- other variables similar as in previous case

Simple problems I

Terzaghi's 1D problem

- confined soil sample placed in container with liquid
- bottom side impermeable, top fully drained and subjected to constant vertical stress
- due to symmetry the problem can be solved in 1D
- analytical solution by Terzaghi (1923) in the form of infinite series



- applied stress induces sudden increase of pressure in the sample
- after consolidation, the pressure drops to the external level
- due to low permeability, consolidation takes certain time

Simple problems II

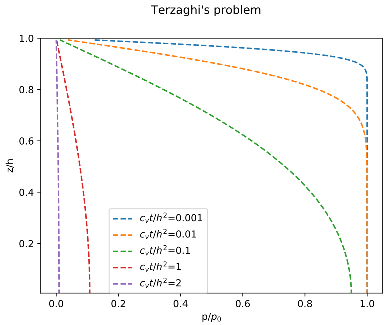
The 1D Biot problem can be reduced to a scalar parabolic equation

$$\partial_t p = c_v \partial^2_{xx} p, \qquad c_v = \frac{k}{S + \frac{\alpha^2}{K + \frac{4}{3}G}}$$

The quantity $c_v t/h^2$ indicates whether the system is consolidated or not. In this problem, for

$$\frac{c_v t}{h^2} > 2$$

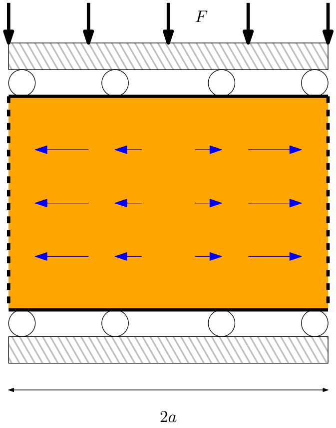
the pressure is almost constant.



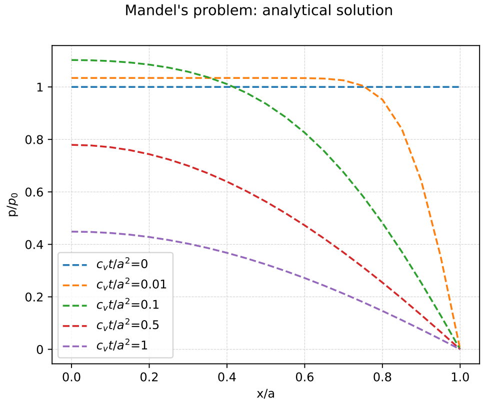
Simple problems III

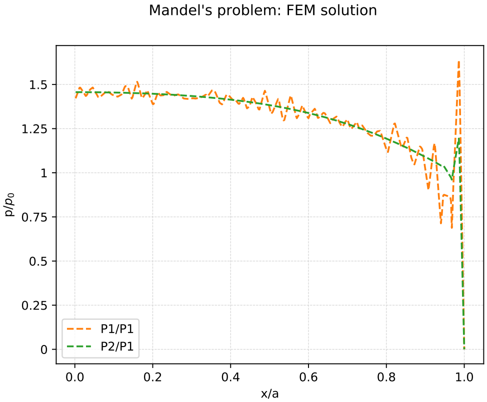
Mandel's problem

- rectangular sample subjected to constant vertical stress
- lateral sides drained, top and bottom impermeable
- semi-analytical solution by Mandel (1963) in the form of infinite series, depending on roots of a nonlinear equation
- after consolidation period, pressure drops to exterior pressure
- due to low permeability and sudden pressure drop on lateral sides, pressure temporarily increases inside the domain (Mandel-Cryer effect)



Simple problems IV



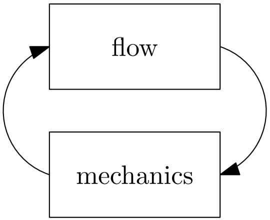


Left: Analytical solution of pressure, right: comparison of FEM solutions.

Iterative splitting

Iterative splittings

- Biot problem is fully coupled
- Monolithic solution is unconditionally stable, but large problems need suitable preconditioners
- Splitting of mechanics and flow can be advantageous but requires careful design



Iterative splittings of skew-symmetric problems I

The discretized two- and three-field formulation of Biot system has skew-symmetric saddle-point structure:

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix}$$

Block Gauss-Seidel method (BGS):

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{0} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^{i+1} \\ \boldsymbol{x}_2^{i+1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^{i} \\ \boldsymbol{x}_2^{i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix}$$

It can be shown that BGS converges only if $\boldsymbol{C} \succ \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^\top \Rightarrow$ conditional convergence.

Iterative splittings of skew-symmetric problems II

This gives the **fixed-strain splitting**:

1. Given $\boldsymbol{u}^i$, find p^{i+1}:

$$\partial_t \left(S p^{i+1} + \alpha \operatorname{div} \boldsymbol{u}^i \right) - \operatorname{div}(k \nabla p^{i+1}) = b$$

2. Given p^{i+1}, find $\boldsymbol{u}^{i+1}$:

$$-\operatorname{div}(\boldsymbol{C} \varepsilon(\boldsymbol{u}^{i+1})) + \alpha \nabla p^{i+1} = \boldsymbol{f}$$

Theorem (Conditional convergence of fixed-strain splitting)

The fixed-strain splitting method is convergenct under the condition

$$S > \frac{\alpha^2}{K}.$$

Iterative splittings of skew-symmetric problems III

Proof:

- differences $(\delta_p^{i+1}, \delta_{\boldsymbol{u}}^i) := (p^{i+1} - p^i, \boldsymbol{u}^i - \boldsymbol{u}^{i-1})$ satisfy:

$$\partial_t \left(S\delta_p^{i+1} + \alpha \operatorname{div} \delta_{\boldsymbol{u}}^i\right) - \operatorname{div}(k\nabla \delta p^{i+1}) = 0$$

$$-\operatorname{div}(\boldsymbol{C}\varepsilon(\delta_{\boldsymbol{u}}^i)) + \alpha\nabla\delta p^i = \boldsymbol{0}$$

- multiply by $\partial_t \delta_p^{i+1}$ / differentiate and multiply by $\partial_t \delta_{\boldsymbol{u}}^i$ and integrate:

$$S\left\|\partial_t \delta_p^{i+1}\right\|_2^2 + k\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\delta_p^{i+1}\right\|_2^2 + K\left\|\operatorname{div}\partial_t \delta_{\boldsymbol{u}}^i\right\|_2^2 + \underbrace{\alpha(\partial_t \delta_p^{i+1}, \operatorname{div}\partial_t \delta_{\boldsymbol{u}}^i)}_{\leqslant \frac{\alpha^2}{2K}\left\|\partial_t \delta_p^{i+1}\right\|_2^2 + \frac{K}{2}\left\|\operatorname{div}\partial_t \delta_{\boldsymbol{u}}^i\right\|_2^2} \leqslant \underbrace{\alpha(\partial_t \delta_p^{i}, \operatorname{div}\partial_t \delta_{\boldsymbol{u}}^i)}_{\leqslant \frac{\alpha^2}{2K}\left\|\partial_t \delta_p^{i}\right\|_2^2 + \frac{K}{2}\left\|\operatorname{div}\partial_t \delta_{\boldsymbol{u}}^i\right\|_2^2}$$

Iterative splittings of skew-symmetric problems IV

- result:

$$\left(S - \frac{\alpha^2}{2K}\right) \left\| \partial_t \, \delta_p^{i+1} \right\|_2^2 + k \frac{\mathrm{d}}{\mathrm{d}t} \left\| \nabla \delta_p^{i+1} \right\|_2^2 \leqslant \frac{\alpha^2}{2K} \left\| \partial_t \, \delta_p^{i} \right\|_2^2$$

- convergence if

$$S - \frac{\alpha^2}{2K} > \frac{\alpha^2}{2K} \quad \Leftrightarrow \quad S > \frac{\alpha^2}{K}$$

□

Iterative splittings of skew-symmetric problems V

Stabilized BGS: Schur complement approach

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 - \boldsymbol{x}_1^i \\ \boldsymbol{x}_2 - \boldsymbol{x}_2^i \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^i \end{bmatrix}$$

Block LU factorization gives ($\boldsymbol{S} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^\top$):

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{B}\boldsymbol{A}^{-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{0} & \boldsymbol{C} + \boldsymbol{S} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 - \boldsymbol{x}_1^i \\ \boldsymbol{x}_2 - \boldsymbol{x}_2^i \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^i \end{bmatrix}$$

This leads to an iterative scheme with approximate Schur complement $\widetilde{\boldsymbol{S}}$:

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{B}\boldsymbol{A}^{-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{0} & \boldsymbol{C} + \widetilde{\boldsymbol{S}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^{i+1} - \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^{i+1} - \boldsymbol{x}_2^i \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^i \end{bmatrix}$$

Iterative splittings of skew-symmetric problems VI

Applying the inverse of the first matrix we get:

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{0} & \boldsymbol{C}+\widetilde{\boldsymbol{S}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^{i+1} - \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^{i+1} - \boldsymbol{x}_2^i \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^i \end{bmatrix}$$

This can be rewritten as a stabilized BGS:

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \widetilde{\boldsymbol{S}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^{i+1} - \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^{i+1} - \boldsymbol{x}_2^i \end{bmatrix} + \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{0} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^{i+1} \\ \boldsymbol{x}_2^{i+1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^i \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix}$$

or alternatively as a preconditioned Richardson method:

$$\begin{bmatrix} \boldsymbol{x}_1^{i+1} \\ \boldsymbol{x}_2^{i+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^i \end{bmatrix} + \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{0} & \boldsymbol{C}+\widetilde{\boldsymbol{S}} \end{bmatrix}^{-1} \left(\begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^i \\ \boldsymbol{x}_2^i \end{bmatrix} \right)$$

Iterative splittings of skew-symmetric problems VII

Application on two-field Biot system:

- Approximate Schur complement: scaled L^2-product $\beta(p,q)$
- Result: **Fixed-stress splitting**
  1. Given $(\boldsymbol{u}^i, p^i)$, find p^{i+1}:

$$\beta\partial_t\left(p^{i+1}-p^i\right)+\partial_t\left(Sp^{i+1}+\alpha\operatorname{div}\boldsymbol{u}^i\right)-\operatorname{div}(k\nabla p^{i+1})=b$$

  2. Given p^{i+1}, find $\boldsymbol{u}^{i+1}$:

$$-\operatorname{div}(\boldsymbol{C}\varepsilon(\boldsymbol{u}^{i+1}))+\alpha\nabla p^{i+1}=\boldsymbol{f}$$

Theorem (Unconditional convergence of fixed-stress splitting)

The fixed-stress splitting method is convergent if $\beta \geqslant \frac{1}{2}\frac{\alpha^2}{K}$. *Fastest convergence is obtained for* $\beta = \frac{1}{2}\frac{\alpha^2}{K}$.

Iterative splittings of skew-symmetric problems VIII

Proof:

- differences $(\delta_p^{i+1}, \delta_{\boldsymbol{u}}^i)$ satisfy:

$$(S+\beta)\partial_t\,\delta_p^{i+1} - \operatorname{div}(k\nabla\delta p^{i+1}) + \alpha\operatorname{div}\partial_t\,\delta_{\boldsymbol{u}}^i = \beta\partial_t\,\delta_p^i$$

$$-\operatorname{div}(\boldsymbol{C}\varepsilon(\delta_{\boldsymbol{u}}^i)) + \alpha\nabla\delta p^i = \boldsymbol{0}$$

- multiply by $\partial_t\,\delta_p^{i+1}$ / differentiate and multiply by $\partial_t\,\delta_{\boldsymbol{u}}^i$ and integrate:

$$(S+\beta)\left\|\partial_t\,\delta_p^{i+1}\right\|_2^2 + k\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\delta_p^{i+1}\right\|_2^2 + K\left\|\operatorname{div}\partial_t\,\delta_{\boldsymbol{u}}^i\right\|_2^2$$

$$+\,\alpha(\partial_t\,\delta_p^{i+1}, \operatorname{div}\partial_t\,\delta_{\boldsymbol{u}}^i) \leqslant (\partial_t\,\delta_p^i, \underbrace{\beta\partial_t\,\delta_p^{i+1} + \alpha\operatorname{div}\partial_t\,\delta_{\boldsymbol{u}}^i}_{=:\sigma})$$

$$\leqslant \frac{\beta}{2}\left\|\partial_t\,\delta_p^i\right\|_2^2 + \frac{1}{2\beta}\|\sigma\|_2^2$$

Iterative splittings of skew-symmetric problems IX

- polarization identity:

$$\alpha(\partial_t \delta_p^{i+1}, \operatorname{div} \partial_t \delta_{\boldsymbol{u}}^i) = \frac{1}{2\beta} \|\sigma\|_2^2 - \frac{\beta}{2} \left\|\partial_t \delta_p^{i+1}\right\|_2^2 - \frac{\alpha^2}{2\beta} \left\|\operatorname{div} \partial_t \delta_{\boldsymbol{u}}^i\right\|_2^2$$

- result:

$$\left(S + \frac{\beta}{2}\right) \left\|\partial_t \delta_p^{i+1}\right\|_2^2 + k \frac{\mathrm{d}}{\mathrm{d}t} \left\|\nabla \delta_p^{i+1}\right\|_2^2 + \frac{1}{2\beta} \|\sigma\|_2^2 + \left(K - \frac{\alpha^2}{2\beta}\right) \left\|\operatorname{div} \partial_t \delta_{\boldsymbol{u}}^i\right\|_2^2 \leqslant \frac{\beta}{2} \left\|\partial_t \delta_p^i\right\|_2^2 + \frac{1}{2\beta} \|\sigma\|_2^2$$

- convergence if

$$K - \frac{\alpha^2}{2\beta} \geqslant 0 \qquad \Leftrightarrow \qquad \beta \geqslant \frac{\alpha^2}{2K}$$

□

Iterative splittings of skew-symmetric problems X

One can also switch the order of elasticity and flow

- Approximate Schur complement: scaled divergence $\gamma \operatorname{div}(\boldsymbol{u}^{i+1} - \boldsymbol{u}^{i})$
- Resulting scheme: **undrained splitting**

1. Given $(\boldsymbol{u}^{i}, p^{i})$, find $\boldsymbol{u}^{i+1}$:

$$-\operatorname{div}(\gamma \operatorname{div}(\boldsymbol{u}^{i+1} - \boldsymbol{u}^{i})\boldsymbol{I} + \boldsymbol{C}\varepsilon(\boldsymbol{u}^{i+1})) + \alpha \nabla p^{i} = \boldsymbol{f}$$

2. Given $\boldsymbol{u}^{i+1}$, find p^{i+1}:

$$\partial_t \left(S p^{i+1} + \alpha \operatorname{div} \boldsymbol{u}^{i+1}\right) - \operatorname{div}(k \nabla p^{i+1}) = b$$

Theorem (Unconditional convergence of undrained splitting)

The undrained splitting method is convergent for $\gamma \geqslant \frac{\alpha^2}{2S}$.

Iterative splittings of skew-symmetric problems XI

Remarks:

- Stable schemes: fixed-stress and undrained split
- Fixed-stress split generally has faster convergence
- Undrained split adds constraints on discretization of elasticity to avoid spurious oscillations
- References: Mikelić and Wheeler (2013), White et al. (2016), Both et al. (2017)

Iterative splitting of symmetric problems I

The discretized five-field formulation of Biot system has symmetric saddle-point structure:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix}$$

Since the matrix is blockwise s.p.d., the BGS is equivalent to alternating minimization

$$\boldsymbol{x}_1^i \to \boldsymbol{x}_2^i := \operatorname*{argmin}_{\boldsymbol{y}} J(\boldsymbol{x}_1^i, \boldsymbol{y}), \qquad \boldsymbol{x}_2^i \to \boldsymbol{x}_1^{i+1} := \operatorname*{argmin}_{\boldsymbol{y}} J(\boldsymbol{y}, \boldsymbol{x}_2^i)$$

of the quadratic functional

$$J(\boldsymbol{x}_1, \boldsymbol{x}_2) = \left(\frac{1}{2} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix} \right) \cdot \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}$$

This method is unconditionally stable.

Iterative splitting of symmetric problems II

Resulting scheme: **fixed-stress splitting**

1. Given σ^i, find $(p^{i+1}, \boldsymbol{q}^{i+1})$:

$$\partial_t \left(S p^{i+1} + \alpha \operatorname{tr}(\boldsymbol{C}^{-1}(\sigma^i + \alpha p^{i+1} \boldsymbol{I}))\right) + \operatorname{div} \boldsymbol{q}^{i+1} = b$$
$$k^{-1} \boldsymbol{q}^{i+1} + \nabla p^{i+1} = \boldsymbol{0}$$

2. Given p^{i+1}, find $(\sigma^{i+1}, \boldsymbol{r}^{i+1}, \boldsymbol{u}^{i+1})$:

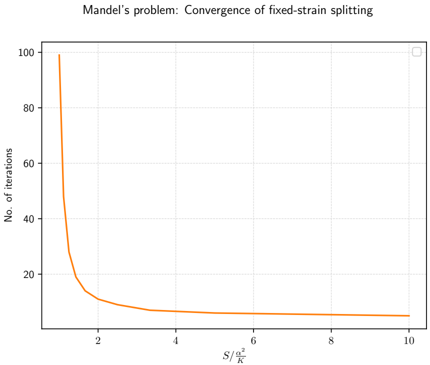
$$-\operatorname{div} \sigma^{i+1} = \boldsymbol{f}$$
$$\boldsymbol{C}^{-1}(\sigma^{i+1} + \alpha p^{i+1} \boldsymbol{I}) - \nabla \boldsymbol{u}^{i+1} + \operatorname{as}^* \boldsymbol{r}^{i+1} = \boldsymbol{0}$$
$$\operatorname{as} \sigma^{i+1} = \boldsymbol{0}$$

Theorem

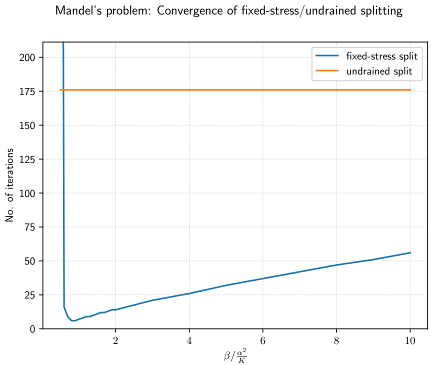
The fixed-stress splitting method for five-field formulation is convergent.

Convergence of iterative schemes: example

Mandel's problem, implicit Euler, P2/P1 FEM



Conditional convergence of fixed-strain splitting.



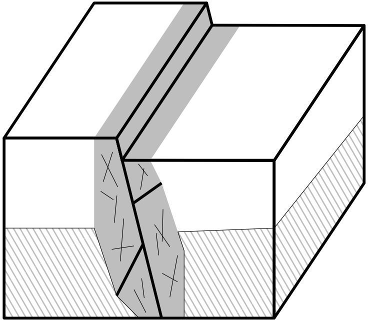
Convergence of fixed-stress splitting and undrained splitting.

Applications in rock hydro-mechanics

Modelling heterogeneities in rock hydro-mechanics I

Two types of rock heterogeneity:

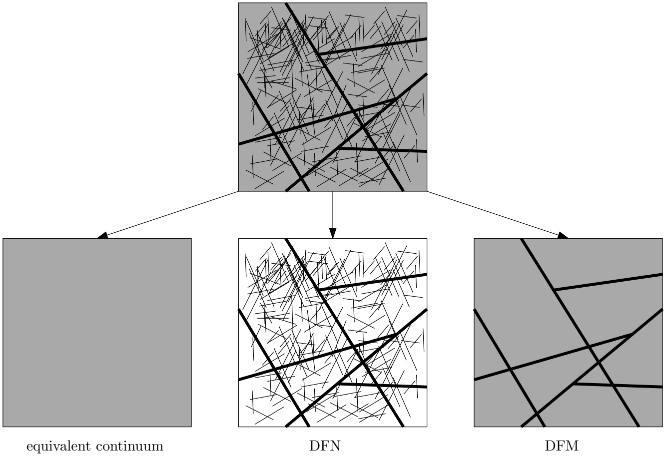
- local variations in bulk properties
- macroscopic fractures and fault zones with narrow width but large size



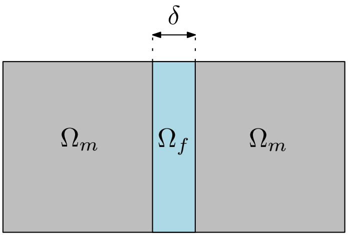
Modelling heterogeneities in rock hydro-mechanics II

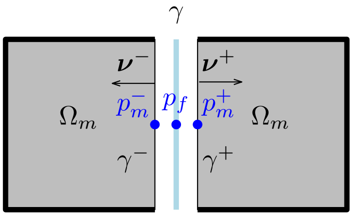
Modelling approaches:

- equivalent continuum
- discrete fracture network (DFN)
- discrete fracture-matrix (DFM)



Discrete fracture-matrix models





- fields $(\boldsymbol{u}, p, \dots)$ in $\Omega_m \cup \Omega_f \longrightarrow$ fields $(\boldsymbol{u}_m, p_m, \dots)$ in Ω_m and $(\boldsymbol{u}_f, p_f, \dots)$ in γ
- semi-discrete operators:

$$\widetilde{\nabla} p = \nabla_t p_f + \nabla_\nu p, \quad \nabla_\nu p = \frac{1}{2}\left(\Delta^+ p \boldsymbol{\nu}^+ + \Delta^- p \boldsymbol{\nu}^-\right), \quad \Delta^\pm p = \frac{2}{\delta}\left(p_m^\pm - p_f\right)$$

$$\widetilde{\operatorname{div}}\, \boldsymbol{u} = \operatorname{div}_t \boldsymbol{u}_f + \operatorname{div}_\nu \boldsymbol{u}, \qquad \operatorname{div}_\nu \boldsymbol{u} = \frac{1}{2}\left(\Delta^+ \boldsymbol{u} \cdot \boldsymbol{\nu}^+ + \Delta^- \boldsymbol{u} \cdot \boldsymbol{\nu}^-\right)$$

DFM model of poroelasticity

1. Biot equations in rock matrix Ω_m:

$$-\operatorname{div}\boldsymbol{\sigma}_m + \alpha\nabla p_m = \boldsymbol{f}_m, \quad \boldsymbol{\sigma}_m = \boldsymbol{C}\varepsilon(\boldsymbol{u}_m)$$
$$\partial_t\left(Sp_m + \alpha\operatorname{div}\boldsymbol{u}_m\right) + \operatorname{div}\boldsymbol{q}_m = b_m, \quad \boldsymbol{q}_m = k\nabla p_m$$

2. Biot equations in fracture γ:

$$-\widetilde{\operatorname{div}}\,\boldsymbol{\sigma} + \alpha\widetilde{\nabla}p = \boldsymbol{f}_f, \quad \boldsymbol{\sigma}_f = \tfrac{1}{2}\boldsymbol{C}(\widetilde{\nabla}\boldsymbol{u} + \widetilde{\nabla}\boldsymbol{u}^\top)$$
$$\partial_t\left(Sp_f + \alpha\widetilde{\operatorname{div}}\,\boldsymbol{u}\right) + \widetilde{\operatorname{div}}\,\boldsymbol{q} = b_f, \quad \boldsymbol{q}_f = k\widetilde{\nabla}p$$

3. Continuity of flux and tangential traction on fracture-matrix interface:

$$\boldsymbol{q}_m^\pm\cdot\boldsymbol{\nu}^\pm = \boldsymbol{q}_f^\pm\cdot\boldsymbol{\nu}^\pm \qquad (\boldsymbol{\sigma}_m^\pm\boldsymbol{\nu}^\pm)_t = (\boldsymbol{\sigma}_f^\pm\boldsymbol{\nu}^\pm)_t$$

4. Constraints on minimal fracture aperture:

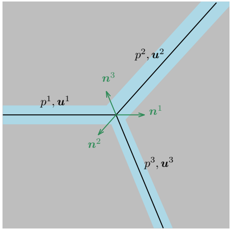
$$\delta + (\boldsymbol{u}_m^+\cdot\boldsymbol{\nu}^+ + \boldsymbol{u}_m^-\cdot\boldsymbol{\nu}^-) \geqslant \delta_{min} \qquad (\Delta^+\boldsymbol{\sigma} + \Delta^-\boldsymbol{\sigma})\boldsymbol{\nu}\cdot\boldsymbol{\nu} \geqslant 0$$

Discretization of DFM model I

- Generalization to immersed fractures, crossings and branching:

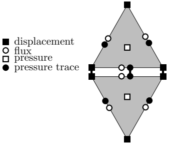
$$\sum_i \sigma_f^i \boldsymbol{n}^i = \boldsymbol{0}$$

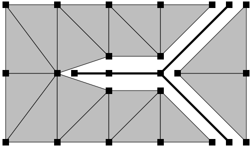
$$\sum_i \boldsymbol{q}_f^i \cdot \boldsymbol{n}^i = 0$$



- Compatible discretization of domain and fracture
- FE spaces: P1/MH

displacement	P_1
pressure	P_0
flux	RT_0
pressure trace	P_0 on edges





Discretization of DFM model II

- algebraic form of fixed-stress splitting scheme:

$$\min_{\boldsymbol{u}_i^m} \left(\frac{1}{2} \boldsymbol{A} \boldsymbol{u}_i^m - \boldsymbol{f}^m - \boldsymbol{B}^\top \boldsymbol{u}_i^f \right) \cdot \boldsymbol{u}_i^m, \quad \boldsymbol{E} \boldsymbol{u}_i^m \leqslant \boldsymbol{c}$$

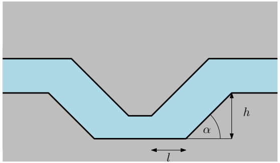
$$(\boldsymbol{C} + \beta \widetilde{\boldsymbol{S}}) \boldsymbol{u}_{i+1}^f = \boldsymbol{f}^f + \beta \widetilde{\boldsymbol{S}} \boldsymbol{u}_i^f - \boldsymbol{B} \boldsymbol{u}_i^m$$

- contact problems solved using quadratic programming (PERMON)
- implementation: Flow123d

Joint work with J. Kružík, D. Horák

Nonlinear fracture couplings

- Contact with shear dilation



$$\delta_{min} = \delta_{min}(\boldsymbol{u}) \ldots \text{ bounded, Lipschitz continuous}$$

Solution by successive approximations:

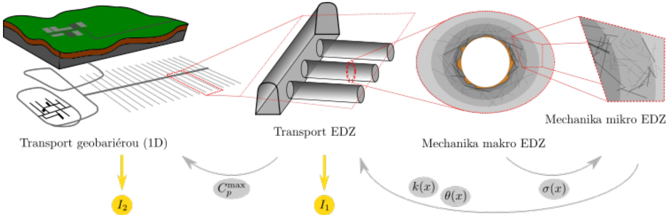
$$\mathcal{P} : \delta_{min} \mapsto \boldsymbol{u}; \qquad \boldsymbol{u}^{n+1} := \mathcal{P}(\delta_{min}(\boldsymbol{u}^n))$$

- Cubic law for fracture permeability

$$k_f = \frac{(\delta + \boldsymbol{u}^+ \cdot \boldsymbol{\nu}^+ + \boldsymbol{u}^- \cdot \boldsymbol{\nu}^-)^2}{12\mu}$$

Application: Homogenization of conductivity in EDZ I

Multiscale model of excavation damage zone

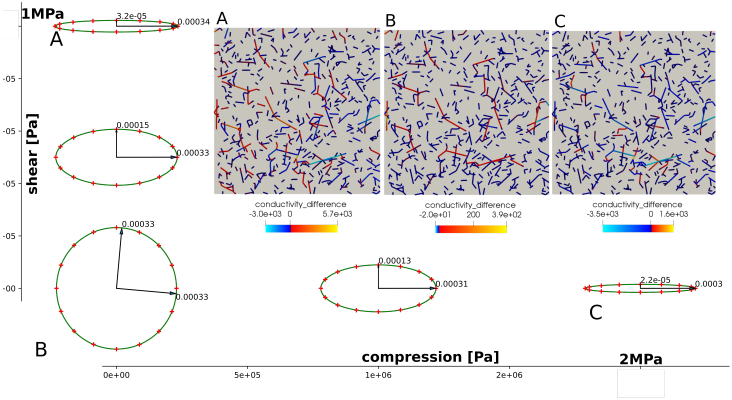


Computation of equivalent hydraulic conductivity of a local DFM model:

- series of computations with prescribed pressure gradient
- least squares fitting of conductivity tensor from averaged velocity and given pressure gradient
- influence of stress/deformation

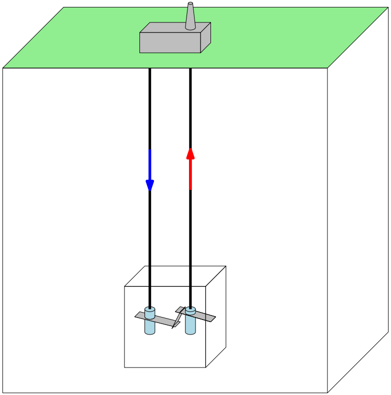
Joint work with J. Březina, M. Špetlík

Application: Homogenization of conductivity in EDZ II



Application: Geothermal system I

- Thermo-hydraulic model of geothermal heat exchanger
- Hydro-mechanical model to describe stimulation (opening) of preexisting fractures
- Fractures with high permeability and low stiffness
- Simulation of power during 30 years of operation
- Comparison for
  - no stimulation
  - stimulation by nonlinear HM model with fracture contact and cubic law



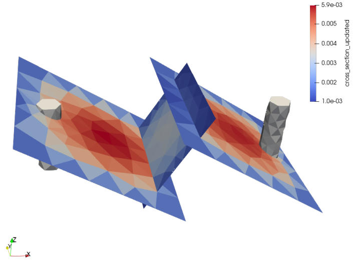
rock: granite
depth: 5 km
distance of wells: 200 m
open part of wells: 100 m
computational domain: cube 600 m \
cylinders $\varnothing = 10\ m$ around wells

Joint work with J. Březina, P. Exner

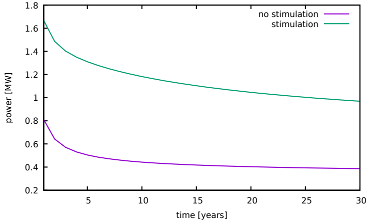
Application: Geothermal system II

- Fractures:
  initial cross-section: 1 mm
  initial conductivity: $k_f/k_m = 10^5$
  Young modulus: $E_f/E_m = 10^{-9}$

1. **HM model of hydraulic stimulation**
   injection pressure: 10 MPa
   initial cross-section: 1 mm
2. **TH model of heat production**
   injection pressure: 1 MPa
   bottom temperature: 150°C
   input water temperature: 15°C

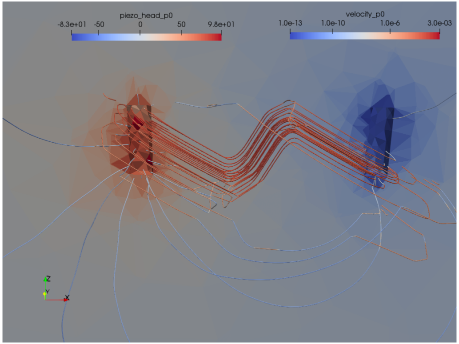


Cross-section of stimulated fractures

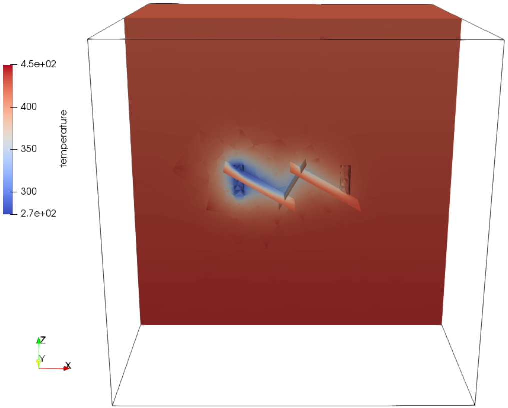


Power (with and without stimulation)

Application: Geothermal system III



Piezometric head and velocity streamlines



Temperature after 30 years

Conclusion

Biot system of poroelasticity

- equivalent formulations using primal/dual variables
- FEM/FVM approximations and their stability
- iterative splittings and their stability
- DFM models for fractured rocks
- real-world applications

Thank you for attention!

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