On a Solvability of Contact Problems with Visco-Plastic Friction in the Thermo-Visco-Plastic Bingham Rheology

Jiří Nedoma [a]

[a] *Institute of Computer Science AS CR, Prague*

Abstract

This paper deals with the solvability of contact problems with a local visco-plastic friction in the thermo-visco-plastic Bingham rheology. The generalized case of bodies of arbitrary shapes being in mutual contacts is investigated. The model problem represents mathematical models of the Earth's mantle movements, of the volcanic zones, etc. Numerical approaches in the dynamic case, based on the semi-implicit scheme in time and a finite element approximation in the space, and in the stationary flow case, based on the penalization, regularization and finite element techniques and semi-implicit scheme in thermal part of the problem, are shortly developed and discussed.

Key words: Thermo-visco-plastic Bingham rheology, contact problems with friction, variational inequalities, Earth's mantle movements, radioactive waste repositories

1 Introduction

The important mathematical problems in the present geodynamics and geomechanics are simulations of the geodynamical processes in the Earth's interior (see [2,5,12,15,17], etc.). There may be giant circulation cells in the mantle of the Earth. The mantle rock materials are strongly heated from below and are transported into the upper parts of the Earth below the lithosphere [2,12,15], etc. This mass transport can be described by the boundary value problems of thermo-visco-plastic fluid with visco-plastic friction through the $3D$ channels in a bounded domain in $\mathbb{R}^3$. Moreover, some geodynamical processes connected with heat flow and diffusion involving phase-change phenomena give rise to free boundary problems for parabolic partial differential equations of the multiphase problems [11,12,14].

Preprint submitted to Elsevier Science *7 September 2004*

In general, the global model of geodynamical processes inside the Earth is described by the system of partial differential equations describing the stress-strain rate field, the geothermal field, the magnetic field, the gravity field as well as local processes taking place in local areas inside the Earth, like the phase changes of the first and the second orders (i.e. solidification, melting, recrystallization, metallization, changes of magnetic properties of rocks, etc.) [9–14]. We see that such global models are very complicated. The model presented in this paper also describes geodynamical processes in the rift and subduction zones and, therefore, it can be applied for simulations of geodynamical and geomechanical processes in such special areas as the San Andreas fault zone and the radioactive waste repositories.

In the paper the constitutive relations of the thermo-Bingham rheology, the friction laws on the contact boundaries, an existence theorem and some ideas how to solve the problem numerically, will be presented.

2 Formulation of the Problem

2.1 Rheology and the constitutive relations

The Bingham rheology has the following property: the rock material starts to flow if and only if the applied forces exceed a certain limit, the so-called yield limit. We speak also about Bingham solid/fluid media.

Let $\mathbf{u} = (u_i)$ be the material velocity, let $D = (D_{ij})$ denote the strain rate tensor and $D^D = (D^D_{ij})$ the strain rate deviator defined by

$$D_{ij} = D_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad D^D_{ij} = D_{ij} - \frac{1}{3} D_{kk}\delta_{ij}\,, \tag{1}$$

where δ_{ij} is the Kronecker symbol. Denote by ${}^B\tau = ({}^B\tau_{ij})$ the Cauchy stress tensor and its deviator ${}^B\tau^D = ({}^B\tau^D_{ij})$ by

$${}^B\tau^D_{ij} = {}^B\tau_{ij} - \frac{1}{3}\,{}^B\tau_{kk}\delta_{ij} \quad \text{i.e.} \quad {}^B\tau^D = {}^B\tau + pI_3\,, \tag{2}$$

where $-p$ denotes the spherical part of the stress tensor and has a meaning of the pressure and I_3 is the identity tensor. Besides the deviators D^D_{ij} and ${}^B\tau^D_{ij}$ another tensor deviator $S^D = (S^D_{ij})$ can be introduced, which corresponds to the plastic properties of the rock material and which will be defined below.

In the rheology by a process we mean a collection of sufficiently smooth functions $t \to D^D_{ij}(t)$, $t \to {}^B\tau^D_{ij}(t)$, $t \to S^D_{ij}(t)$ for $t \in [0, t_p]$, where t_p is the duration of the rheological process in the medium. In the Bingham rheology for any process we find

$$
\begin{aligned}
{}^B\tau^D &= S^D + 2\hat{\mu} D^D \,, && (3)\\
f(S^D) &= S_{II} - \hat{g}^2 = \frac{1}{2}|S^D|^2 - \hat{g}^2 \le 0 && (4)
\end{aligned}
$$

the so-called von Mises relation or the yield condition,

$$
D^D = 2\lambda S^D \,, \tag{5}
$$

where $S_{II} = \frac{1}{2} S^D_{ij} S^D_{ij} = \frac{1}{2}|S^D|^2$, $|S^D|^2 = S^D_{ij} S^D_{ij}$, $\hat{\mu} > 0$ is the threshold of viscosity, $\hat{g}$ the threshold of plasticity or the yield limit, $\hat{g}/\sqrt{2}$ is the yield stress in pure shear and λ is a scalar function defined by

$$
\begin{aligned}
&\text{if } f(S^D) < 0 \quad \text{or} \quad \left(f(S^D) = 0 \ \text{ and } \ \frac{\partial f(S^D(t))}{\partial t} < 0 \right) \ \text{ then } \ \lambda(t) = 0 \,,\\
&\text{if } f(S^D) = 0 \ \text{ and } \ \frac{\partial f(S^D(t))}{\partial t} = 0 \ \text{ then } \ \lambda(t) > 0 \,. \qquad (6)
\end{aligned}
$$

The von Mises relation yields that the invariant S_{II} cannot reach the square of the yield limit. Relations (5) and (6) yield that the deviator of the strain rate tensor can change in the case if S^D_{ij} rests on the surface $f(S^D) = 0$, moving along it. In other cases $D^D_{ij} = 0$, which represents the argument why the relation $S^{1/2}_{II} = \hat{g}$ is called the yield condition.

Moreover, (3) and (5) yield

$$
{}^B\tau^D_{ij} = (1 + 4\hat{\mu}\lambda) S^D_{ij}, \quad |{}^B\tau^D| = (1 + 4\hat{\mu}\lambda)|S^D|, \quad {}^B\tau^{1/2}_{II} = (1 + 4\hat{\mu}\lambda) S^{1/2}_{II} \,, \tag{7}
$$

where $|{}^B\tau^D|^2 = {}^B\tau^D_{ij}\,{}^B\tau^D_{ij}$, ${}^B\tau_{II} = \frac{1}{2}\,{}^B\tau^D_{ij}\,{}^B\tau^D_{ij} = \frac{1}{2}|{}^B\tau^D|^2$ is an invariant of the stress tensor in the Bingham rheology.

Let ${}^B\tau^{1/2}_{II} = 2^{-\frac{1}{2}}\,|{}^B\tau^D| > \hat{g}$ then from (7c) and (4) we find $\lambda > 0$ and from (6) we have $S^{1/2}_{II} = 2^{-\frac{1}{2}}\,|S^D| = \hat{g} > 0$. Then (7c) yields

$$
\lambda = (4\hat{\mu})^{-1} \left({}^B\tau^{1/2}_{II} \hat{g}^{-1} - 1 \right) . \tag{8}
$$

Hence and from (5) and (7a,c) we obtain

$$D^D_{ij} = (2\hat{\mu})^{-1}(1 - \hat{g}\,{}^B\tau_{II}^{-1/2})\,{}^B\tau^D_{ij} = (2\hat{\mu})^{-1}\left(1 - 2^{\frac{1}{2}}\hat{g}|\,{}^B\tau^D|^{-1}\right)\,{}^B\tau^D_{ij}\,. \quad (9)$$

For incompressible rocks, for which $D_{kk} = 0$, and any duration of the rheological process $t_p > 0$, using (1) we find

$$D_{ij} = (2\hat{\mu})^{-1}(1 - \hat{g}\,{}^B\tau_{II}^{-1/2})\,{}^B\tau^D_{ij} = (2\hat{\mu})^{-1}(1 - 2^{\frac{1}{2}}\hat{g}\,|\,{}^B\tau^D|^{-1})\,{}^B\tau^D_{ij}\,. \quad (10)$$

Assume that ${}^B\tau_{II}^{1/2} \leq \hat{g}$. Then if $S_{II}^{1/2} = \hat{g}$ from (7c) we find that $\lambda = 0$ and if $S_{II}^{1/2} < \hat{g}$, then (6) yields also $\lambda = 0$. Hence, by using (5) and the incompressibility condition, i.e. $D_{kk} = 0$, we have $D_{ij} = 0$.

Summarizing, the constitutive law in the Bingham rheology can be written as

$$\begin{aligned} D_{ij} &= (2\hat{\mu})^{-1}(1 - \hat{g}\,{}^B\tau_{II}^{-1/2})\,{}^B\tau^D_{ij} \quad \text{if} \quad {}^B\tau_{II}^{1/2} > \hat{g}\,, \\ D_{ij} &= 0 \quad \text{if} \quad {}^B\tau_{II}^{1/2} \leq \hat{g}\,. \end{aligned} \quad (11)$$

In order to invert relations (11) we introduce an invariant of the strain rate tensor by $D_{II} = \frac{1}{2}D_{ij}D_{ij} = \frac{1}{2}|D|^2$, $|D|^2 = D_{ij}D_{ij}$. Let $|D| = 0$, then from (11) we find ${}^B\tau_{II}^{1/2} \leq \hat{g}$; if $|D| \neq 0$, then from (11) we obtain ${}^B\tau_{II}^{1/2} > \hat{g}$ and ${}^B\tau_{II}^{1/2} = \hat{g} + 2\hat{\mu}D_{II}^{1/2}$. Hence, from the incompressibility condition $D_{kk} = 0$, (2) and (10) we find

$${}^B\tau_{ij} = -p\delta_{ij} + \hat{g}D_{ij}D_{II}^{-1/2} + 2\hat{\mu}D_{ij}\,, \quad (12)$$

representing the constitutive stress-strain rate relation in the Bingham rheology. The relation (12) makes sense only when $D_{II} \neq 0$. For $\hat{g} = 0$, we have a classical viscous incompressible (Newtonian) fluid; for small $\hat{g} > 0$, we have strongly visco-plastic materials close to the classical viscous fluid and for $\hat{g} \to \infty$, we have absolutely rigid rock materials. If $\hat{g}$ is strictly positive, we can observe rigid zones inside the flow (representing e.g. very important property for mushy zones); when $\hat{g}$ further increases, these rigid zones become larger and larger (representing very important property for solidification) and start completely rigid when $\hat{g}$ is sufficiently large. In this case, when these rigid zones block the flow, we speak about blocking property of the Bingham fluid. From these point of view the Bingham rheology is useful for solidification, recrystallization and melting processes inside the Earth's mantle and in volcanic areas. This problem represents an open mathematical problem.

The thermal stresses are defined by the well-known relation

$$ {}^{T}\tau_{ij} = -\beta_{ij}(T - T_0)\,, \tag{13} $$

where β_{ij} is the coefficient of thermal expansion, T, T_0 are the actual and initial temperatures.

The stress-strain rate relation in thermo-Bingham rheology is as follows:

$$ \tau_{ij} = {}^{B}\tau_{ij} + {}^{T}\tau_{ij}\,. \tag{14} $$

2.2 The problem formulation and the friction law

Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, $\Omega = \bigcup_{\iota=1}^{r} \Omega^{\iota}$, be a union of bounded domains occupied by visco-plastic bodies with a smooth boundary $\partial\Omega = \Gamma_u \cup \Gamma_\tau \cup \Gamma_0 \cup \Gamma_c$, where Γ_u represents one part of the boundary, where the velocity is prescribed; Γ_τ is the part of the boundary, where the loading is prescribed; Γ_0 is the part of the boundary, where the bilateral contact condition is given and $\Gamma_c\Big(= \bigcup_{k,l} \Gamma_c^{kl}, \Gamma_c^{kl} = \partial\Omega^k \cap \partial\Omega^l,\ k \neq l,\ k, l \in \{1, \ldots, r\}\Big)$ represents the contact boundary. Assume the Eulerian coordinate system as a spatial variable system. Any repeated index implies summation from 1 to N. Let $t \in I \equiv (0, t_p)$, $t_p > 0$, where t_p is the duration of all rheological processes. Let $u_n = \mathbf{u} \cdot \mathbf{n}$, $\mathbf{u}_t = \mathbf{u} - u_n\mathbf{n}$, $\boldsymbol{\tau} = (\tau_{ij}n_j)$, $\tau_n = \tau_{ij}n_jn_i$, $\boldsymbol{\tau}_t = \boldsymbol{\tau} - \tau_n\mathbf{n}$ be the normal and tangential components of the velocity and stress vectors.

Next, we will solve the following problem:

Problem $(\mathcal{P})$**:** Find a pair of functions $(T, \mathbf{u}) : \Omega \times I \to \mathbb{R} \times \mathbb{R}^N$, $N = 2, 3$, and a stress tensor $\tau_{ij} : \Omega \times I \to \mathbb{R}^{N\times N}$ satisfying

$$\rho\left(\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}\right)=\frac{\partial}{\partial x_j}\tau_{ij}+f_i \quad \text{in} \quad \Omega\times I\,; \tag{15}$$

$$\operatorname{div}\mathbf{u}=0 \quad \text{in} \quad \Omega\times I\,; \tag{16}$$

$$\rho c_e\left(\frac{\partial T}{\partial t}+u_k\frac{\partial T}{\partial x_k}\right)-\rho\beta_{ij}T_0D_{ij}(\mathbf{u})=\frac{\partial}{\partial x_j}\left(\kappa_{ij}\frac{\partial T}{\partial x_i}\right)+W \text{ in } \Omega\times I; \tag{17}$$

$$\tau_{ij}=-p\delta_{ij}+\hat{g}D_{ij}D_{II}^{-1/2}+2\hat{\mu}D_{ij}-\beta_{ij}(T-T_0)\,; \tag{18}$$

$$D_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\;;$$

provided $D_{II}\neq 0$, ${}^{B}\tau_{II}^{1/2}\leq\hat{g}$ if $D_{II}=0$, with the boundary value and contact conditions

$$T(\mathbf{x},t)=T_1(\mathbf{x},t)\,,\quad \tau_{ij}n_j=P_i \quad \text{on} \quad \Gamma_\tau\times I\,, \tag{19}$$

$$\kappa_{ij}\frac{\partial T(\mathbf{x},t)}{\partial x_j}n_j=0\,,\quad \mathbf{u}(\mathbf{x},t)=\mathbf{u}_1(\mathbf{x},t) \quad \text{on} \quad \Gamma_u\times I\,, \tag{20}$$

$$\kappa_{ij}\frac{\partial T(\mathbf{x},t)}{\partial x_j}n_j=q(\mathbf{x},t)\,,\quad u_n(\mathbf{x},t)=0\,,\quad \boldsymbol{\tau}_t(\mathbf{x},t)=0 \text{ on } \Gamma_0\times I\,, \tag{21}$$

$$\begin{gathered}T^k(\mathbf{x},t)=T^l(\mathbf{x},t)\,,\quad \text{and}\\ \kappa_{ij}\frac{\partial T(\mathbf{x},t)}{\partial x_j}n_{j|(k)}=\kappa_{ij}\frac{\partial T(\mathbf{x},t)}{\partial x_j}n_{j|(l)} \quad \text{on} \quad \Gamma_c\times I\,,\end{gathered} \tag{22}$$

and the bilateral contact condition with local friction law of the form [7]

$$\begin{aligned}&u_n^k-u_n^l=0 \quad \text{and} \quad |\boldsymbol{\tau}_t^{kl}|\leq\mathcal{F}_c^{kl}S^{Dkl}\,,\\ &\text{if } |\boldsymbol{\tau}_t^{kl}|<\mathcal{F}_c^{kl}S^{Dkl} \quad \text{then} \quad \mathbf{u}_t^k-\mathbf{u}_t^l=0\,,\\ &\text{if } |\boldsymbol{\tau}_t^{kl}|=\mathcal{F}_c^{kl}S^{Dkl} \quad \text{then there exists} \quad \lambda\geq 0\\ &\qquad\qquad \text{such that} \quad \mathbf{u}_t^k-\mathbf{u}_t^l=-\lambda\boldsymbol{\tau}_t^{kl}\,,\end{aligned} \tag{23}$$

and the initial conditions

$$T(\mathbf{x},t_0)=T_0(\mathbf{x})\,,\quad \mathbf{u}(\mathbf{x},t_0)=0\,, \tag{24}$$

where the positive direction of the normal $\mathbf{n}$ to Γ_c^{kl} is related to the region Ω^k, $\mathcal{F}_c^{kl}$ is a coefficient of friction, $u_n^k=u_i^kn_i^k$, $u_n^l=u_i^ln_i^l=-u_i^ln_i^k$ and $\boldsymbol{\tau}_t^{kl}\equiv\boldsymbol{\tau}_t^k=\boldsymbol{\tau}_t^l$ and where ρ is a density, c_e – a specific heat, κ_{ij} – a thermal conductivity, W – thermal sources, q – a heat flow, $\mathbf{P}$ – surface forces, $\mathbf{f}$ – body forces, T_0, T_1, $\mathbf{u}_1$ are given functions.

If $S^D = |\tau_n|$, then (23) is the classical Coulombian law of friction. Such model problems describe the geodynamical processes in upper parts of the Earth. The processes in the lower mantle and in the volcanic areas, connected with transportation of heated light materials from the mantle/core boundary through the GTLV channels below the lithosphere or the heated viscous materials from the volcanic chambers through the deep faults upto the Earth's surface, respectively, can be described by a visco-plastic friction law. Setting (see e.g. [3]) $S^D = |{}^B\tau^D|$ and if we determine $|{}^B\tau^D|$ from (12) we obtain $|{}^B\tau^D| = 2^{\frac{1}{2}}\hat{g} + 2\hat{\mu}|D(\mathbf{u})|$. We see that the contact condition (23) depends on the solution of the investigated problem.

For simplicity we will assume that $\mathbf{u}_1(\mathbf{x},t) = 0$, $T_1(\mathbf{x},t) = 0$.

3 Variational (Weak) Solution of the Problem

We will introduce the Sobolev spaces of vector-functions having generalized derivatives of the (possibly fractional) order s of the type $[H^s(\Omega)]^k \equiv H^{s,k}(\Omega)$, where $H^s(\Omega) \equiv W_2^s(\Omega)$. The norm will be denoted by $\|\cdot\|_{s,k}$ and the scalar product by $(\cdot,\cdot)_s$ (for each integer k). We set $H^{0,k}(\Omega) \equiv L^{2,k}(\Omega)$. We introduce the space $C_0^\infty(\Omega)$ as a space of all functions in $C^\infty(\Omega)$ with a compact support in Ω^ι, $\iota = 1,\dots,r$. The space is equipped with the ordinary countable system of seminorms and as usual $C_0^\infty(\Omega,\mathbb{R}^N) = [C_0^\infty(\Omega)]^N$. We introduce the space $L^p(\Omega)$, $1 \le p \le \infty$, as the space of all measurable functions such that $\|f\|_{L^p(\Omega)} = (\int_\Omega |f(\mathbf{x})|^p d\mathbf{x})^{1/p} < +\infty$. By $L^\infty(\Omega)$ we denote the set of all measurable functions almost everywhere on Ω such that $\|f\|_\infty = \operatorname{ess\,sup}_\Omega |f(\mathbf{x})|$ is finite. Moreover, we define for $s \ge 1$ the following spaces and sets:

$$\begin{aligned}
{}^1H^{s,N}(\Omega) &= \sqcap_{\iota=1}^r {}^1H^{s,N}(\Omega^\iota) = \{\mathbf{v}|\mathbf{v} \in H^{s,N}(\Omega),\ \operatorname{div}\mathbf{v} = 0 \quad \text{in} \quad \cup_{\iota=1}^r \Omega^\iota\,, \\
&\qquad v_n^k - v_n^l = 0 \quad \text{on} \quad \cup_{k,l}\Gamma_c^{kl},\ v_n|_{\Gamma_0} = 0\}\,, \\
{}^2H^{s,1}(\Omega) &= \sqcap_{\iota=1}^r {}^2H^{s,1}(\Omega^\iota) = \{z|z \in H^{s,1}(\Omega),\ z^k = z^l \quad \text{on} \quad \cup_{k,l}\Gamma_c^{kl},\}\,, \\
{}^1V_s(\Omega) &= \{\mathbf{v}|\mathbf{v} \in {}^1H^{s,N}(\Omega), \mathbf{v}|_{\Gamma_u} = 0\} \\
{}^2V_s(\Omega) &= \{z|z \in {}^2H^{s,1}(\Omega),\ z|_{\Gamma_\tau} = 0\}\,.
\end{aligned}$$

Then ${}^1H^{s,N}(\Omega)$ is a Hilbert space with the norm $\|\cdot\|_{s,N}$, ${}^2H^{s,1}(\Omega)$ is a Hilbert space with the norm $\|\cdot\|_{s,1}$. For the sake of simplicity we put ${}^1H^{1,N}(\Omega) = {}^1H(\Omega)$, $\|\cdot\|_{1,N} \equiv \|\cdot\|_1$ and in $H^{0,N}(\Omega)$, $\|\cdot\|_{0,N} \equiv \|\cdot\|_0$. Let us put ${}^1\mathcal{H}(\Omega) = {}^1H^{1,N}(\Omega) \cap C_0^\infty(\Omega,\mathbb{R}^N)$, ${}^2\mathcal{H}(\Omega) = {}^2H^{1,1}(\Omega) \cap C_0^\infty(\Omega)$. We will denote the dual space of ${}^1H^{s,N}(\Omega)$ by $({}^1H^{s,N}(\Omega))'$ and similarly in other cases. For $s > 1$ ${}^iV_s \subset {}^iV_1 = {}^iV \subset {}^iV_0 \subset {}^iV' \subset {}^iV_s'$, $i = 1,2$. Furthermore, we put $\mathbf{v}' = \frac{\partial \mathbf{v}}{\partial t}$, $z' = \frac{\partial z}{\partial t}$,

$$ {}^1H = \{\mathbf{v}|\mathbf{v} \in L^2(I;\, {}^1V_s),\, \mathbf{v}' \in L^2(I;\, {}^1V_0(\Omega)),\, \mathbf{v}(\mathbf{x}, t_0) = 0\}\,, $$
$$ {}^2H = \{z|z \in L^2(I;\, {}^2V_s),\, z' \in L^2(I;\, {}^2V_0(\Omega)),\, z(\mathbf{x}, t_0) = T_0(\mathbf{x})\}\,. $$

We will denote by $\mathbf{w}_j$ eigenfunctions of a canonical isomorphism ${}^1\Lambda_s$: ${}^1H^{s,N} \to ({}^1H^{s,N})'$, i.e. $(\mathbf{w}, \mathbf{v})_s = \lambda_i(\mathbf{w}_j, \mathbf{v})_0 \;\forall \mathbf{v} \in {}^1H^{s,N}$, $\|\mathbf{w}_j\|_0 = 1$ and similarly we define $z_j \in {}^2H^{s,1}$.

Throughtout the paper we will assume that $\mathbf{f}^\iota(\mathbf{x}, t) \in L^2(I; ({}^1H^{1,N}(\Omega^\iota))')$, $\mathbf{P}(\mathbf{x}, t) \in L^2(I; L^{2,N}(\Gamma_\tau))$, $\partial\beta^\iota_{ij}(\mathbf{x})/\partial x_j \in L^\infty(\Omega^\iota)$, $\forall i, j \in \{1, \ldots, N\}$, $q \in L^2(I; L^2(\Gamma_0))$, $T^\iota_0(\mathbf{x}) \in {}^2H^{1,1}(\Omega^\iota)$, ρ^ι, $\hat{g}^\iota$, $\hat{\mu}^\iota$ are piecewise constant and positive, $W^\iota \in L^2(I; ({}^2H^{1,1}(\Omega^\iota))')$, $\kappa^\iota_{ij} \in L^\infty(\Omega^\iota)$ are Lipschitz on Ω^ι, and satisfy the usual symmetry condition $\kappa^\iota_{ij} = \kappa^\iota_{ji}$ and $\kappa^\iota_{ij}\xi_i\xi_j \geq c_T\|\xi\|^2_1$, $\mathbf{x} \in \Omega^\iota$, $\xi \in \mathbb{R}^N$, $c_T = \text{const.} > 0$, $\mathcal{F}^{kl}_c \in L^\infty(\Gamma^{kl}_c)$, $\mathcal{F}^{kl}_c \geq 0$ a.e. on $\bigcup_{k,l} \Gamma^{kl}_c$.

For $\mathbf{u}, \mathbf{v} \in H^{1,N}(\Omega)$, T, $z \in H^{1,1}(\Omega)$ we put

$$ \begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \sum_{\iota=1}^{r} a^\iota(\mathbf{u}^\iota, \mathbf{v}^\iota) = 2\int_\Omega \hat{\mu} D_{ij}(\mathbf{u}) D_{ij}(\mathbf{v})\, d\mathbf{x}\,, \\ (\mathbf{u}', \mathbf{v}) &= \sum_{\iota=1}^{r} (\mathbf{u}^{\iota\prime}, \mathbf{v}^\iota) = \int_\Omega \rho \mathbf{u}'\mathbf{v}\, d\mathbf{x}\,, \\ (T', z) &= \sum_{\iota=1}^{r} (T^{\iota\prime}, z^\iota) = \int_\Omega \rho c_e T' z\, d\mathbf{x}\,, \\ a_T(T, z) &= \sum_{\iota=1}^{r} a^\iota_T(T^\iota, z^\iota) = \int_\Omega \kappa_{ij} \frac{\partial T}{\partial x_i}\frac{\partial z}{\partial x_j}\, d\mathbf{x}\,, \\ S(\mathbf{v}) &= \sum_{\iota=1}^{r} S^\iota(\mathbf{v}^\iota) = \int_\Omega f_i v_i\, d\mathbf{x} + \int_{\Gamma_\tau} P_i v_i\, ds \equiv (\mathbf{F}, \mathbf{v})\,, \\ s(z) &= \sum_{\iota=1}^{r} s^\iota(z^\iota) = \int_\Omega W z\, d\mathbf{x} + \int_{\Gamma_0} q z\, ds \equiv (Q, z)\,, \\ b_0(\mathbf{v}, g, z) &= \sum_{\iota=1}^{r} b^\iota_0(\mathbf{v}^\iota, g^\iota, z^\iota) = \int_\Omega \rho c_e v_k \frac{\partial g}{\partial x_k} z\, d\mathbf{x}\,, \\ b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \sum_{\iota=1}^{r} b^\iota(\mathbf{u}^\iota, \mathbf{v}^\iota, \mathbf{w}^\iota) = \int_\Omega \rho u_i \frac{\partial v_j}{\partial x_i} w_j\, d\mathbf{x}\,, \\ j(\mathbf{v}) &= \sum_{\iota=1}^{r} j^\iota(\mathbf{v}^\iota) = 2\int_\Omega \hat{g}(D_{II}(\mathbf{v}))^{\frac{1}{2}}\, d\mathbf{x}\,, \\ j_g(\mathbf{v}) &= \sum_{\iota=1}^{r} j^\iota_g(\mathbf{v}^\iota) = \int_{\cup_{kl}\Gamma^{kl}_c} \mathcal{F}^{kl}_c |S^{Dkl}|\, |\mathbf{v}^k_t - \mathbf{v}^l_t|\, ds\,, \end{aligned} $$

$$b_s(T,\mathbf{v}) = \sum_{\iota=1}^{r} b_s^\iota(T^\iota,\mathbf{v}^\iota) = \int_\Omega \frac{\partial}{\partial x_j}(\beta_{ij}T)v_i d\mathbf{x}\,,$$

$$b_p(\mathbf{v},z) = \sum_{\iota=1}^{r} b_p^\iota(\mathbf{v}^\iota,z^\iota) = \int_\Omega \rho T_0 \beta_{ij}\frac{\partial v_i}{\partial x_j} z d\mathbf{x}\,.$$

Let us multiply (15) by $\mathbf{v}-\mathbf{u}(t)$ and (17) by $z-T(t)$, respectively, and further add both equations. We integrate the sum over Ω and apply the Green theorem satisfying the boundary conditions. Then after some modification, including among other the integration in time over the interval I, we obtain the following variational (weak) formulation:

Problem $(\mathcal{P})_v$: Find a pair of functions $(T,\mathbf{u})$ such that $T \in {}^2H$, $\mathbf{u} \in {}^1H$ and

$$\begin{aligned}
&\int_I [(\mathbf{u}'(t),\mathbf{v}-\mathbf{u}(t)) + (T'(t), z-T(t)) + \\
&+a(\mathbf{u}(t),\mathbf{v}-\mathbf{u}(t)) + a_T(T(t),z-T(t)) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}-\mathbf{u}(t)) + \\
&+b_0(\mathbf{u}(t),T(t),z-T(t)) + b_s(T(t)-T_0,\mathbf{v}-\mathbf{u}(t)) + \\
&+b_p(\mathbf{u}(t),z-T(t)) + j(\mathbf{v}(t)) - j(\mathbf{u}(t)) + j_g(\mathbf{v}(t)) - j_g(\mathbf{u}(t))]dt \geq \\
&\geq \int_I [S(\mathbf{v}-\mathbf{u}(t)) + s(z-T(t))]dt \quad \forall(\mathbf{v},z) \in {}^1H \times {}^2H
\end{aligned} \tag{25}$$

holds for a.a. $t \in I$.

From the above assumptions $a(\mathbf{u},\mathbf{v}) = a(\mathbf{v},\mathbf{u})$, $a_T(T,z) = a_T(z,T)$ hold. Moreover, they yield that for $\mathbf{u} \in {}^1H^{1,N}(\Omega)$, $T \in {}^2H^{1,1}(\Omega)$ there exist constants $c_B > 0$, $c_T > 0$ such that $a(\mathbf{u},\mathbf{u}) \geq c_B\|\mathbf{u}\|_{1,N}^2$ for all $\mathbf{u} \in {}^1H^{1,N}(\Omega)$, $a_T(T,T) \geq c_T\|T\|_{1,1}^2$ for all $T \in {}^2H^{1,1}(\Omega)$. Furthermore, we have (see [18])

$$\begin{aligned}
&|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leq c_4\|\mathbf{u}\|_{1,N}^{1/2}\|\mathbf{u}\|_{0,N}^{1/2}\|\mathbf{w}\|_{1,N}^{1/2}\|\mathbf{w}\|_{0,N}^{1/2}\|\mathbf{v}\|_{s,N},\ s = \frac{1}{2}N\,,\\
&\qquad \mathbf{u} \in {}^1H^{1,N}(\Omega),\ \mathbf{w} \in H^{s,N}(\Omega)\,,\\
&|b_0(\mathbf{u},y,z)| \leq c_5\|\mathbf{u}\|_{1,N}^{1/2}\|\mathbf{u}\|_{0,N}^{1/2}\|z\|_{1,1}^{1/2}\|z\|_{0,1}^{1/2}\|y\|_{s,1},\ s = \frac{1}{2}N\,,\\
&\qquad \mathbf{u} \in {}^1H^{1,N}(\Omega),\ z \in H^{s,1}(\Omega)\,,\\
&\ b(\mathbf{u},\mathbf{u},\mathbf{u}) = 0,\ b_0(\mathbf{u},z,z) = 0\,,
\end{aligned}$$

since $\mathcal{F}_c^{kl} \in L^\infty(\Gamma_c^{kl})$, $\mathcal{F}_c^{kl} \geq 0$, then $j_g(\mathbf{u}(t)) \geq 0$ and moreover, if $\mathbf{u},\mathbf{v},\mathbf{w} \in {}^1\mathcal{H}(\Omega)$ then $b(\mathbf{u},\mathbf{v},\mathbf{w}) + b(\mathbf{u},\mathbf{w},\mathbf{v}) = 0$, which is valid also for $\mathbf{u} \in L^{2,N}(\Omega)$, $\mathbf{v},\mathbf{w} \in {}^1H^{1,N}(\Omega)$; $b(\mathbf{u},\mathbf{u},\mathbf{v}) = -b(\mathbf{u},\mathbf{v},\mathbf{u})$ for $\mathbf{u},\mathbf{v} \in {}^1\mathcal{H}(\Omega)$. Similarly, $b_0(\mathbf{u},y,z) + b_0(\mathbf{u},z,y) = 0$ for $\mathbf{u} \in {}^1H^{0,N}(\Omega)$, $y,z \in H_0^{1,1}(\Omega)$.

The main result is represented by the following theorem:

Theorem 1 *Let* $N \geq 2$*,* $s = \frac{N}{2}$*. Let* $\mathbf{f} \in L^2(I; ({}^1V_1(\Omega))')$*,* $\mathbf{P} \in L^2(I; L^{2,N}(\Gamma_\tau))$*,* $W \in L^2(I; \, ({}^2V_1(\Omega))')$*,* $\frac{\partial \beta_{ij}}{\partial x_j} \in L^\infty(\Omega)$*,* $\forall i, j \in \{1, \ldots, N\}$*,* $q \in L^2(I; L^2(\Gamma_u))$*,* $T_1 \in L^2(I; L^2(\Gamma_\tau))$*,* $T_0(\mathbf{x}) \in H^{1,1}(\Omega)$*,* $\hat{g}$*,* $\hat{\mu}$ *are piecewise constant,* $\kappa_{ij} \in L^\infty(\Omega)$*,* $\mathcal{F}_c^{kl} \in L^\infty(\Gamma_c^{kl})$*,* $\mathcal{F}_c^{kl} \geq 0$ *a.e. on* $\bigcup_{k,l} \Gamma_c^{kl}$*. Then there exists a pair of functions* $(\mathbf{u}, T)$ *such that*

$$\begin{aligned} \mathbf{u} &\in L^2(I; \, {}^1V_s(\Omega)) \cap L^\infty(I; \, {}^1V_0(\Omega)), \; \mathbf{u}' \in L^2(I; ({}^1V_s(\Omega))')\,, \\ T &\in L^2(I; \, {}^2V_s(\Omega)) \cap L^\infty(I; \, {}^2V_0(\Omega)), \; \mathbf{u}' \in L^2(I; ({}^2V_s(\Omega))')\,, \\ \mathbf{u}(\mathbf{x}, t_0) &= 0, T(\mathbf{x}, t_0) = T_0(\mathbf{x}) \end{aligned}$$

and satisfying the variational inequality (25).

PROOF. To prove the theorem the triple regulatizations will be used. Let us introduce the regularization of $j(\mathbf{v}(t))$ by

$$j_\varepsilon(\mathbf{v}(t)) = \frac{2}{1+\varepsilon} \int_\Omega \hat{g}(D_{II}(\mathbf{v}(t)))^{(1+\varepsilon)/2} d\mathbf{x}, \quad \varepsilon > 0 \quad \text{and} \quad (j'_\varepsilon(\mathbf{v}), \mathbf{v}) \geq 0\,. \tag{26}$$

Since the functional $j_g(\mathbf{v})$ is not differentiable in the Gâteaux sense, therefore it can be regularized by its regularization $j_{g\varepsilon}(\mathbf{v})$. Let us introduce the function $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined as $\psi_\varepsilon(y) = \sqrt{(y^2 + \varepsilon^2)} - \varepsilon$, regularizing the function $y \to |y|$. The function ψ_ε is differentiable and the following inequality

$$|\,|y| - \psi_\varepsilon(|y|)| < \varepsilon \quad \forall y \in \mathbb{R}, \;\; \varepsilon \geq 0$$

holds. Then the functional $j_g(\mathbf{v})$ will be regularized by its regularization $j_{g\varepsilon}(\mathbf{v})$, defined by

$$j_{g\varepsilon}(\mathbf{v}(t)) = \int_{\cup_{k,l}\Gamma_c^{kl}} \mathcal{F}_c^{kl} |S^{Dkl}| \psi_\varepsilon(|\mathbf{v}_t^k - \mathbf{v}_t^l|) ds \quad \text{and} \quad (j'_{g\varepsilon}(\mathbf{v}), \mathbf{v}) \geq 0\,. \tag{27}$$

The third regularization will be introduced by adding the viscous terms $\eta((\mathbf{u}_{\varepsilon\eta}(t), \mathbf{v}))_s$ and $\eta((T_{\varepsilon\eta}(t), z))_s$, where $s = \frac{N}{2}$ and η is a positive number. For $N = 2$ we obtain $s = 1$ and ${}^1H^{s,N}(\Omega) = {}^1H^{1,N}(\Omega)$, ${}^2H^{s,1}(\Omega) = {}^2H^{1,1}(\Omega)$ and the added viscous terms are of the same order as the bilinear form, and therefore, it can be omitted. We remark that these terms have the physical meaning as the viscosity.

Thus we will to solve the triple regularized problem:

Problem $(\mathcal{P}_r)_v$: Find a pair of functions $(\mathbf{u}_{\varepsilon\eta}, T_{\varepsilon\eta}) \in {}^1H \times {}^2H$ such that

$$\begin{aligned}
&\int_I [(\mathbf{u}'_{\varepsilon\eta}(t), \mathbf{v} - \mathbf{u}_{\varepsilon\eta}(t)) + (T'_{\varepsilon\eta}(t), z - T_{\varepsilon\eta}(t)) + a(\mathbf{u}_{\varepsilon\eta}(t), \mathbf{v} - \mathbf{u}_{\varepsilon\eta}(t)) + \\
&+ a_T(T_{\varepsilon\eta}(t), z - T_{\varepsilon\eta}(t)) + b(\mathbf{u}_{\varepsilon\eta}(t), \mathbf{u}_{\varepsilon\eta}(t), \mathbf{v} - \mathbf{u}_{\varepsilon\eta}(t)) + \eta((\mathbf{u}_{\varepsilon\eta}(t), \mathbf{v} - \\
&- \mathbf{u}_{\varepsilon\eta}(t)))_s + \eta((T_{\varepsilon\eta}(t), z - T_{\varepsilon\eta}(t)))_s + b_0(\mathbf{u}_{\varepsilon\eta}(t), T_{\varepsilon\eta}(t), z - T_{\varepsilon\eta}(t)) + \\
&+ b_p(\mathbf{u}_{\varepsilon\eta}(t), z - T_{\varepsilon\eta}(t)) + b_s(T_{\varepsilon\eta}(t) - T_0, \mathbf{v} - \mathbf{u}_{\varepsilon\eta}(t)) + j(\mathbf{v}(t)) - \\
&- j(\mathbf{u}_{\varepsilon\eta}(t)) + j_g(\mathbf{v}(t)) - j_g(\mathbf{u}_{\varepsilon\eta}(t))]dt \geq \\
&\geq \int_I [S(\mathbf{v} - \mathbf{u}_{\varepsilon\eta}(t)) + s(z - T_{\varepsilon\eta}(t))]dt \quad \forall (\mathbf{v}, z) \in {}^1H \times {}^2H\,. \qquad (28)
\end{aligned}$$

The method of the proof is similar to that of Theorem 3 in [9] and it is as follows:

(1) the existence of the solution of (28) will be based on the Galerkin approximation technique;
(2) a priori estimates I and II independent of ε and η will be given;
(3) limitation processes for the Galerkin approximation (i.e. over m) and for $\varepsilon \to 0$, $\eta \to 0$ will be performed;
(4) the uniqueness of the solution of (25) for $N = 2$ can be proved only.

The existence of a pair of functions $(\mathbf{u}_{\varepsilon\eta}, T_{\varepsilon\eta})$ will be proved by means of the finite-dimensional approximation. The proof is similar of that of Theorem 3 in [9]. We construct a countable bases of the spaces ${}^1V_s(\Omega)$ and ${}^2V_s(\Omega)$, i.e. each finite subsets are linearly independent and span$\{\mathbf{v}_i | i = 1, 2, \dots\}$, span$\{z_i | i = 1, 2, \dots\}$ are dense in ${}^1V_s(\Omega)$ and ${}^2V_s(\Omega)$, respectively, as ${}^1V_s(\Omega)$ and ${}^2V_s(\Omega)$ are separable spaces. Let us construct spaces spanned by $\{\mathbf{v}_i | 1 \leq j, k \leq m\}$, $\{z_i | 1 \leq j, k \leq m\}$. Then the approximate solution $(\mathbf{u}_m, T_m)$ of the order m satisfies

$$\begin{aligned}
&(\mathbf{u}'_m(t), \mathbf{v}_j) + a(\mathbf{u}_m(t), \mathbf{v}_j) + b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{v}_j) + \eta((\mathbf{u}_m(t), \mathbf{v}_j))_s + \\
&+ b_s(T_m(t) - T_0, \mathbf{v}_j) + (j'_\varepsilon(\mathbf{u}_m(t), \mathbf{v}_j) + (j'_{g\varepsilon}(\mathbf{u}_m(t), \mathbf{v}_j) = S(\mathbf{v}_j), \\
&\hspace{8cm} 1 \leq j \leq m\,. \\
&(T'_m(t), z_j) + a_T(T_m(t), z_j) + b_0(\mathbf{u}_m(t), T_m(t), z_j) + \eta((T_m(t), z_j))_s + \\
&+ b_p(\mathbf{u}_m(t), z_j) = s(z_j) \quad 1 \leq j \leq m\,, \qquad (29)
\end{aligned}$$

$$\mathbf{u}_m(\mathbf{x}, t_0) = 0 \quad T(\mathbf{x}, t_0) = T_0(\mathbf{x})\,. \qquad (30)$$

Since $\{\mathbf{v}_j\}_{j=1}^m$, $\{z_j\}_{j=1}^m$ are linearly independent, the system (29), (30) is the regular system of ordinary differential equations of the first order, and therefore (29), (30) uniquely define $(\mathbf{u}_m, T_m)$ on the interval $I_m = \langle t_0, t_m \rangle$. Therefore, (29) is valid for every test function $\mathbf{v}(t) = \sum_{i=1}^{m} c_i(t)\mathbf{v}_i$, $t \in I_m$, and $z = \sum_{i=1}^{m} d_i(t) z_j$, $t \in I_m$, where c_i, d_i are continuously differentiable functions on I_m, $i = 1, \ldots, m$.

A priori estimate I:

Using assumptions, relations and estimates mentioned above, we have

$$
\begin{aligned}
& b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{u}_m(t)) = 0, \quad b_0(\mathbf{u}_m(t), T_m(t), T_m(t)) = 0 \\
& |b_s((T_m(t) - T_0), \mathbf{u}_m(t) + b_p(\mathbf{u}_m(t), T_m(t))| \leq \\
& \leq c(1 + \|T_m(t)\|_{1,1} \|\mathbf{u}_m(t)\|_{0,N} + \|T_m(t)\|_{0,1} \|\mathbf{u}_m(t)\|_{1,N}) \,. \qquad (31)
\end{aligned}
$$

Via the integration of (29), with $\mathbf{v}_j(t) = \mathbf{u}_m(t)$, $z_j(t) = T_m(t)$, in time over $I_m = (t_0, t_m)$, and since $(j'_\varepsilon(\mathbf{v}), \mathbf{v}) \geq 0$, $(j'_{g\varepsilon}(\mathbf{v}), \mathbf{v}) \geq 0$, using the ellipticity of bilinear forms $a(\mathbf{u}, \mathbf{u})$, $a_T(T, T)$, due to (31), after some modifications as well as applying the Gronwall lemma and then after some more algebra, we find the following estimates

$$
\|\mathbf{u}_m(t)\|_{0,N} \leq c, \ t \in I, \ \int_I \|\mathbf{u}_m(\tau)\|^2_{1,N} d\tau \leq c, \ \eta \int_I \|\mathbf{u}_m(\tau)\|^2_{s,N} d\tau \leq c \quad (32)
$$

$$
\|T_m(t)\|_{0,1} \leq c, \ t \in I, \ \int_I \|T_m(\tau)\|^2_{1,1} d\tau \leq c, \ \eta \int_I \|T_m(\tau)\|^2_{s,1} d\tau \leq c \,. \quad (33)
$$

From these estimates we obtain
$\{\mathbf{u}_m(t), m \in \mathbb{N}\}$ is a bounded subset in $L^2(I; {}^1H)$,
$\{\eta^{1/2}\mathbf{u}_m(t), m \in \mathbb{N}\}$ is a bounded subset in $L^2(I; {}^1V)$,
$\{T_m(t), m \in \mathbb{N}\}$ is a bounded subset in $L^2(I; {}^2H)$,
$\{\eta^{1/2}\mathbf{u}_m(t), m \in \mathbb{N}\}$ is a bounded subset in $L^2(I; {}^2V)$.

To prove a priori estimate II, then similarly as in [4,8,9,12] the system (29), (30) is equivalent to the following system

$$
\begin{aligned}
& (\mathbf{u}'_m + A_B \mathbf{u}_m + \eta \Lambda_s \mathbf{u}_m + j'_\varepsilon(\mathbf{u}_m) + j'_{g\varepsilon}(\mathbf{u}_m) + h_m - \mathbf{F}, \mathbf{v}_j) = 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \leq j \leq m \,, \qquad (34) \\
& (T'_m + A_T T_m + \eta \mathcal{T}_s T_m + g_m - Q, z_j) = 0, \quad 1 \leq j \leq m \,, \qquad (35)
\end{aligned}
$$

with (30), where $a(\mathbf{u}, \mathbf{v}) = (A_B \mathbf{u}, \mathbf{v})$, $A_B \in \mathcal{L}({}^1V_1, {}^1V'_1)$, $a_T(T, z) = a_T(A_T T, z)$, $A_T \in \mathcal{L}({}^2V_1, {}^2V'_1)$, $((\mathbf{u}, \mathbf{v}))_s = (\Lambda_s \mathbf{u}, \mathbf{v})$, $\Lambda_s \in \mathcal{L}({}^1V_s, {}^1V'_s)$, $((T, z))_s = (\mathcal{T}_s T, z)$, $\mathcal{T}_s \in \mathcal{L}({}^2V_s, {}^2V'_s)$, $b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}) + b_s(T_m, \mathbf{v}) = (h_m, \mathbf{v})$, $h_m \in {}^1K_m \subset L^2(I; {}^1V'_s)$, $b_0(\mathbf{u}_m, T_m, z) + b_p(\mathbf{u}_m, z) = (g_m, z)$, $g_m \in {}^2K_m \subset L^2(I; {}^2V'_s)$. Then applying the technique of orthogonal projection and using the technique of [4] we find that

$\mathbf{u}'_m$ is a bounded subset of $L^2(I;\ {}^1V'_s)$,
T'_m is a bounded subset of $L^2(I;\ {}^2V'_s)$. (36)

The limit over m (Galerkin), i.e. the convergence of the finite-dimensional approximation for ε, η being fixed and the limitation over $\varepsilon, \eta \to 0$ finish the existence of the solution $(\mathbf{u}(t), T(t))$ satisfying (25). Uniqueness of the problem for $N = 3$ is an open problem.

4 Numerical Approach

The investigated variational problem (25) may be solved numerically in its dynamical and stationary flow (i.e. when the mass movements are uniform) formulations. Next the ideas of numerical solutions will be shortly discussed only.

(A) Dynamic case:

Let Ω_h be a polyhedral approximation to Ω in $\mathbb{R}^3$ and let its boundary be denoted as $\partial\Omega_h = \Gamma_{uh} \cup \Gamma_{\tau h} \cup \Gamma_{0h} \cup \Gamma_{ch}$. Let $\mathfrak{T}_h$ be a partition of $\overline{\Omega}_h$ by tetrahedra $\mathcal{T}_h$. Let $h = h(\mathcal{T}_h)$ be the maximum diameter of tetrahedral elements $\mathcal{T}_h$. Let $\mathcal{T}_h \in \mathfrak{T}_h$ be a tetrahedron with vertices P_i, $i = 1, \ldots, 4$ and let R_i be the barycentres with respect to the points P_i, $i = 1, \ldots, 4$. Assume that $\{\mathfrak{T}_h\}$ is a regular family of partitioning $\mathfrak{T}_h$ of Ω_h such that $\overline{\Omega}_h = \bigcup_{\mathcal{T}_h \in \mathfrak{T}_h} \mathcal{T}_h$.

Now let us introduce the main idea how to solve the problem $(\mathcal{P})$. Let n be an integer and set $k = t_p/n$. Let

$$
\begin{aligned}
\mathbf{f}_k^{i+\Theta} &= \frac{1}{k}\int_{ik}^{(i+\Theta)k} \mathbf{f}(t)dt,\ i = 0, \ldots, n-1,\ 0 \le \Theta \le 1,\ \mathbf{f}_k^{i+\Theta} \in {}^1V'_s\,,\\
W_k^{i+\Theta} &= \frac{1}{k}\int_{ik}^{(i+\Theta)k} W(t)dt,\ i = 0, \ldots, n-1,\ 0 \le \Theta \le 1,\ W_k^{i+\Theta} \in {}^2V'_s\,.
\end{aligned}
$$

We start with the initial data $\mathbf{u}^0 = 0$, $T^0 = T_0$. When $(\mathbf{u}^0, T^0), \ldots, (\mathbf{u}^i, T^i)$ are known, we define $(\mathbf{u}^{i+1}, T^{i+1})$ as an element of ${}^1V_s \times {}^2V_s$. The existence of a pair of functions $(\mathbf{u}^{i+1}, T^{i+1})$ for each fixed k and each $i \ge 0$ can be proved. As regards the spatial discretization, there are several classes of possibilities for approximation ${}^1V_s$ by its finite element space ${}^1V_h$ [6,12,18]. In the present paper we use the spaces of linear non-conforming finite elements ${}^1V_h$ in the visco-plastic part of the problem and of linear conforming finite elements ${}^2V_h$ in the thermal part of the problem. Let ${}^1\mathcal{V}_h = \{\mathbf{v}_h | v_{hi} \in P_1^*,\ i = 1, \ldots, N,$ $\forall \mathcal{T}_h \in \mathfrak{T}_h$, continuous in barycentres of tetrahedra B_j, $j = 1, \ldots, m$, and equal to zero in B_j laying on Γ_{uh}, $\sum_{i=1,\dots,N} \partial v_{hi}/\partial x_i = 0$, $\mathbf{v}_h = 0$, $\mathbf{x} \in \Omega\backslash\overline{\Omega}_h\}$, where P_1^* denotes the space of all non-conforming linear polynomials and m is a number of barycentres of tetrahedra, ${}^1V_h = \{\mathbf{v}_h|\mathbf{v}_h \in {}^1\mathcal{V}_h,\ \mathbf{v}_h = \mathbf{u}_1 \text{ on } \Gamma_{uh}\}$ and similarly, let ${}^2\mathcal{V}_h$, ${}^2V_h$ be the spaces of conforming linear finite elements.

Given a "triangulation" $\mathcal{T}_h$ of Ω_h, we assume $(\mathbf{u}_h^0, T_h^0)$ to be given in ${}^1V_h \times {}^2V_h$, and, taken so that $(\mathbf{u}_h^0, T_h^0)$ tends to $(0, T_0)$ in ${}^1V_h \times {}^2V_h$ when $h \to 0_+$. Moreover, we will assume that $\|\mathbf{u}_h^0\|$, $\|T_h^0\|$ are bounded. We define recursively for each k and h a family of elements $(\mathbf{u}_h^0, T_h^0), \dots, (\mathbf{u}_h^i, T_h^i)$ of ${}^1V_h \times {}^2V_h$, which can be based on implicit, semi-implicit and/or explicit schemes. In our paper the semi-implicit scheme will be introduced and shortly discussed. The time derivatives will be approximated by the backward differences. Since $\frac{\mathbf{u}_h^{i+1}-\mathbf{u}_h^i}{k} = \frac{\mathbf{u}_h^{i+\Theta}-\mathbf{u}_h^i}{\Theta k}$, $\frac{T_h^{i+1}-T_h^i}{k} = \frac{T_h^{i+\Theta}-T_h^i}{\Theta k}$, $0 \le \Theta \le 1$, then

$$\begin{aligned}
\mathbf{u}_h^{i+\Theta} &= \Theta\mathbf{u}_h^{i+1} + (1-\Theta)\mathbf{u}_h^i,\ T_h^{i+\Theta} = \Theta T_h^{i+1} + (1-\Theta)T_h^i,\ 0 \le \Theta \le 1\,,\\
\mathbf{u}_h^{i+1} &= \Theta^{-1}\mathbf{u}_h^{i+\Theta} - \frac{(1-\Theta)}{\Theta}\mathbf{u}_h^i,\ T_h^{i+1} = \Theta_h^{-1}T_h^{i+\Theta} - \frac{(1-\Theta)}{\Theta}T_h^i,\ 0 \le \Theta \le 1\,.
\end{aligned}$$

Scheme$(\mathcal{P}_{si})$: When $(\mathbf{u}_h^0, T_h^0), \dots, (\mathbf{u}_h^i, T_h^i)$ are known, then $(\mathbf{u}_h^{i+1}, T_h^{i+1})$ is a solution in ${}^1V_h \times {}^2V_h$ of

$$\begin{aligned}
&k^{-1}(\mathbf{u}_h^{i+1} - \mathbf{u}_h^i, \mathbf{v}_h - \mathbf{u}_h^{i+\Theta}) + a_h(\mathbf{u}_h^{i+\Theta}, \mathbf{v}_h - \mathbf{u}_h^{i+\Theta}) +\\
&+b_h(\mathbf{u}_h^i, \mathbf{u}_h^i, \mathbf{v}_h - \mathbf{u}_h^{i+\Theta}) + b_s(T_h^i - T_h^0, \mathbf{v}_h - \mathbf{u}_h^{i+\Theta}) + j_h(\mathbf{v}_h) - j_h(\mathbf{u}_h^{i+\Theta}) +\\
&+j_{gh}(\mathbf{v}_h) - j_{gh}(\mathbf{u}_h^{i+\Theta}) \ge (\mathbf{F}_h^{i+\Theta}, \mathbf{v}_h - \mathbf{u}_h^{i+\Theta}) \quad \forall \mathbf{v}_h \in {}^1V_h,
\end{aligned} \tag{37}$$

$$\begin{aligned}
&k^{-1}(T_h^{i+1} - T_h^i, z_h - T_h^{i+\Theta}) + a_{Th}(T_h^{i+\Theta}, z_h - T_h^{i+\Theta}) +\\
&+b_{0h}(\mathbf{u}_h^i, T_h^i, z_h - T_h^{i+\Theta}) + b_p(k^{-1}(\mathbf{u}_h^i - \mathbf{u}^{i-1}), z_h - T_h^{i+\Theta}) \ge\\
&\ge (Q_h^{i+\Theta}, z_h - T_h^{i+\Theta}) \quad \forall z_h \in {}^2V_h\,.
\end{aligned} \tag{38}$$

The scheme is valid for $\Theta \ge \frac{1}{2}$; for $\Theta = 1$ we have the semi-implicit scheme and for $\Theta = \frac{1}{2}$ we have the Crank-Nicholson scheme.

Let $|\cdot|_h$ be the norm in $L^2(\Omega)$, $([L^2(\Omega)]^N)$, $\|\cdot\|_h$ in ${}^1\mathcal{V}_h$, ${}^2V_h$ and ${}^2\mathcal{V}_h$, ${}^2V_h$. According to [18] $|\mathbf{u}_h|_h \le d_1\|\mathbf{u}_h\|_h$, $\|\mathbf{u}_h\|_h \le S(h)|\mathbf{u}_h|_h$ $\forall \mathbf{u}_h \in {}^1\mathcal{V}_h$, $|T_h|_h \le d_0\|T_h\|_h$, $\|T_h\|_h \le S_0(h)|T_h|_h$ $\forall T_h \in {}^2\mathcal{V}_h$, d_0, d_1 are independent of h. Furthermore,

$$\begin{aligned}
|b_{0h}(\mathbf{u}_h, T_h, z_h)| &\le c_0|\mathbf{u}_h|_h|T_h|_h\|z_h\|_h \quad \forall \mathbf{u}_h \in {}^1\mathcal{V}_h,\ \forall T_h, z_h \in {}^2\mathcal{V}_h\,,\\
|b_h(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h)| &\le d_1\|\mathbf{u}_h\|_h\|\mathbf{v}_h\|_h\|\mathbf{w}_h\|_h \quad \forall \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in {}^1\mathcal{V}_h\,,
\end{aligned}$$

where c_0 and d_1 do not depend on h,

$$|b_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h)| \le S_1(h)|\mathbf{u}_h|_h\|\mathbf{u}_h\|_h|\mathbf{v}_h|_h \quad \forall \mathbf{u}_h, \mathbf{v}_h \in {}^1\mathcal{V}_h\,,$$

where $S_1(h) \le d_1 S^2(h)$,

$$\begin{aligned} b_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_h) &= 0 \quad \forall \mathbf{u}_h \in {}^1\mathcal{V}_h\,, \\ b_{0h}(\mathbf{u}_h, T_h, z_h) + b_{0h}(\mathbf{u}_h, z_h, T_h) &= 0 \quad \forall T_h, z_h \in {}^2\mathcal{V}_h,\ \forall \mathbf{u}_h \in {}^1\mathcal{V}_h\,, \\ |b_{sh}(T_h - T_0, \mathbf{u}_h) + b_{ph}(\mathbf{u}_h, T_h)| &\le c(1 + \|T_h\|_h|\mathbf{u}_h|_h + |T_h|_h\|\mathbf{u}_h\|_h) \\ &\qquad \forall T_h \in {}^2\mathcal{V}_h, \quad \forall \mathbf{u}_h \in {}^1\mathcal{V}_h\,. \end{aligned}$$

It can be shown that the $(T_h^i, \mathbf{u}_h^i)$ and $(T_h^{i+1}, \mathbf{u}_h^{i+1})$ defined by the scheme $(\mathcal{P}_{si})$ satisfy for $\Theta \ge \frac{1}{2}$ the following conditions:

Theorem 2 *Let the family of "triangulations" $\{\mathfrak{T}_h\}$ be uniformly regular, and let the angles in the tetrahedra be less or equal to $\frac{\pi}{2}$. Let k, h satisfy $kS_0(h) \le d_0$, $kS(h) \le d_1$, where d_0, d_1 are positive constants independent of k, h. Let $\Theta \ge \frac{1}{2}$. Then $(T_h^i, \mathbf{u}_h^i)$ are defined by the scheme $(\mathcal{P}_{si})$ and*

$$|\mathbf{u}_h^i|_h^2 \le c,\ i = 0, \dots, n,\ k\sum_{i=0}^{n-1} \|\mathbf{u}_h^{i+\Theta}\|_h^2 \le c, \quad k\sum_{i=0}^{n-1} |\mathbf{u}_h^{i+1} - \mathbf{u}_h^i|_h^2 \le c, \tag{39}$$

$$|T_h^i|_h^2 \le c_0,\ i = 0, \dots, n,\ k\sum_{i=0}^{n-1} \|T_h^{i+\Theta}\|_h^2 \le c_0, \quad k\sum_{i=0}^{n-1} |T_h^{i+1} - T_h^i|_h^2 \le c_0, \tag{40}$$

hold, where c, c_0 are constants independent of k, h.

The scheme $(\mathcal{P}_{si})$ is stable and convergent. The proofs are similar of that of [6,12]. The difficulty in practical computations is connected with the approximation of the constraint div $\mathbf{v}_h = 0$, i.e. the incompressibility condition of Bingham's fluid. For its approximation see e.g. [6,18]. The studied problem can be also solved by using the penalty and the regularization techniques similarly as in the stationary flow case.

(B) Stationary flow case:

Now let us analyze the case when the mass movements are uniform and $\mathbf{u}_1 \neq 0$, $T_1 = 0$, i.e the boundary value problem (15), (16), (17), (18), (19)-(24a). Then the investigated problem corresponds to the stationary flow of the thermo-Bingham fluid in the region Ω. The Eulerian coordinate will be taken as a spatial variable. The analysis of the thermal part of the problem can be solved by the technique as above. The analysis of the visco-plastic part of the problem will be based on the penalization, regularization and finite element techniques. The algorithm then is parallel to that of [7].

Let us consider

$$^1V_1(\Omega) = \{\mathbf{v}|\mathbf{v} \in {}^1H^{1,N}(\Omega), \mathbf{v}|_{\Gamma_u} = \mathbf{u}_1\}, \quad {}^2V_1(\Omega) = \{z|z \in {}^2H^{1,1}(\Omega), z|_{\Gamma_\tau} = 0\}\,,$$

where ${}^1H^{1,N}(\Omega)$, ${}^2H^{1,1}(\Omega)$ are defined above.

The problem $(\mathcal{P})_v$ leads to the following problem:

Problem $(\mathcal{P}_{sf})_v$**:** Find a pair of functions $(T, \mathbf{u})$, $\mathbf{u} \in {}^1V_1$, $T \in {}^2V_1$ satisfying for every $t \in I \equiv [0, t_p]$, $t_p > 0$,

$$\begin{aligned}
&a(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) + b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) + b_s(T(t) - T_0, \mathbf{v} - \mathbf{u}(t)) + \\
&+j(\mathbf{v}(t)) - j(\mathbf{u}(t)) + j_g(\mathbf{v}(t)) - j_g(\mathbf{u}(t)) \geq S(\mathbf{v} - \mathbf{u}(t)) \quad \forall \mathbf{v} \in {}^1V_1 \\
&(T'(t), z - T(t)) + a_T(T(t), z - T(t)) + b_0(\mathbf{u}(t), T(t), z - T(t)) + \\
&+b_p(\mathbf{u}(t), z - T(t)) \geq s(z - T(t)) \quad \forall z \in {}^2V_1\,,
\end{aligned} \tag{41}$$

$$T(\mathbf{x}, t_0) = T_0(\mathbf{x})\,. \tag{42}$$

In connection with the given data, we suppose that ${}^1V_1 \neq \emptyset$, ${}^2V_1 \neq \emptyset$ and that the physical data satisfy the same conditions as above. Since the algorithm of the thermal part of the problem $(\mathcal{P}_{sf})_v$ is parallel of that of the previous case, we will discussed the idea of the solution of the visco-plastic part of the problem only.

Let us introduce the space $\mathcal{W}$, a closed subspace of $H^{1,N}(\Omega)$, by

$$\mathcal{W} = \left\{\mathbf{v}|\mathbf{v} \in H^{1,N}(\Omega), \mathbf{v}|_{\Gamma_u} = 0, v_n|_{\Gamma_0} = 0, v_n^k - v_n^l = 0 \text{ on } \bigcup_{k,l} \Gamma_c^{kl}\right\}\,, \tag{43}$$

in which the incompressibility condition div $\mathbf{u} = 0$ is not introduced.

Since the linear space $V = {}^1V_1 - \mathbf{u}_1, \mathbf{u}_1 \in {}^1V_1$, on which the variational problem is formulated, contains the condition of incompressibility representing certain cumbersome for numerical solution, therefore we apply a penalty technique, similarly as in the case of incompressible Newtonian fluid (see [18]). The penalty term will be introduced by

$$P(\mathbf{u}_\varepsilon) = \frac{1}{\varepsilon} c(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon),\ c > 0\,,$$

where

$$c(\mathbf{u}, \mathbf{v}) = \int_\Omega (\text{div}\ \mathbf{u})(\text{div}\ \mathbf{v}) d\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v} \in H^{1,N}(\Omega)\,.$$

It can be shown that for each $\varepsilon > 0$ the corresponding penalized variational inequality has a unique solution and that its corresponding solution converges strongly in $H^{1,N}(\Omega)$ to the solution of the initial problem when $\varepsilon \to 0$.

To solve the penalized problem numerically we set $\overline{\mathbf{u}} = \mathbf{u} - \mathbf{u}_1$ and then the finite element technique will be used. Let $\mathcal{W}_h \subset \mathcal{W}$ be a family of finite element subspaces with the property:

$$\forall \mathbf{v} \in \mathcal{W} \text{ there exists } \mathbf{v}_h \in \mathcal{W}_h \text{ such that } \mathbf{v}_h \to \mathbf{v} \text{ in } H^{1,N}(\Omega) \text{ for } h \to 0\,.$$

Then we will solve the following problem:

Problem $(\mathcal{P}_{sf})_h$**:** Find $\overline{\mathbf{u}}_{\varepsilon h} \in \mathcal{W}_h$ satisfying for every $t \in I$ the variational inequality

$$\begin{aligned} & a_h(\overline{\mathbf{u}}_{\varepsilon h}, \mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon h}) + b_h(\overline{\mathbf{u}}_{\varepsilon h}, \overline{\mathbf{u}}_{\varepsilon h}, \mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon h}) + b_{sh}(T_h - T_{0h}, \mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon h}) + \\ & + j_h(\mathbf{v}_h) - j_h(\overline{\mathbf{u}}_{\varepsilon h}) + j_{gh}(\mathbf{v}_h) - j_{gh}(\overline{\mathbf{u}}_{\varepsilon h}) + \frac{1}{\varepsilon} c_h(\overline{\mathbf{u}}_{\varepsilon h}, \mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon h}) \geq \\ & \geq S_h(\mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon h}) \quad \forall \mathbf{v}_h \in \mathcal{W}_h\,. \end{aligned} \tag{44}$$

Lemma 3 *Let $\overline{\mathbf{u}}_{\varepsilon h}$ be a solution of (44) for each $h > 0$ and let $\overline{\mathbf{u}}_\varepsilon$ be the solution of the penalized problem for a fixed $\varepsilon > 0$. Then*

$$\overline{\mathbf{u}}_{\varepsilon h} \to \overline{\mathbf{u}}_\varepsilon \quad \text{strongly in } H^{1,N}(\Omega) \text{ when } h \to 0\,.$$

The proof follows from Lemma A4.2 of [7].

Since the functionals $j(\mathbf{v})$ and $j_g(\mathbf{v})$ are not differentiable in the Gâteaux sense, they can be regularized similarly as above. Let us consider the function $\psi_\gamma : \mathbb{R} \to \mathbb{R}$, which regularizes the function $x \to |x|$, defined by

$$\psi_\gamma(x) = \sqrt{(x^2 + \gamma^2)} - \gamma\,.$$

This function is differentiable and the following inequality $||x| - \psi_\gamma(|x|)| < \gamma$ $\forall x \in R, \gamma > 0$ holds. Then the functionals $j(\mathbf{v})$ and $j_g(\mathbf{v})$ will be regularized by their regularizations $j_\gamma(\mathbf{v})$ and $j_{g\gamma}(\mathbf{v})$, defined by

$$
\begin{aligned}
j_\gamma(\mathbf{v}) &= \int_\Omega 2^{\frac{1}{2}} \hat{g} \psi_\gamma(|D(\mathbf{v} + \mathbf{u}_1)|) d\mathbf{x}\,, \\
j_{g\gamma}(\mathbf{v}) &= \int_{\cup_{kl}\Gamma_c^{kl}} \mathcal{F}_c^{kl} [2^{\frac{1}{2}} \hat{g} + 2\hat{\mu}\psi_\gamma(|D(\mathbf{v} + \mathbf{u}_1)|)] \psi_\gamma(|\mathbf{v}^k - \mathbf{v}^l + (\mathbf{u}_1^k - \mathbf{u}_1^l)|) ds\,.
\end{aligned}
$$

Then we will solve the penalized-regularized problem:

find $\overline{\mathbf{u}}_{\varepsilon h\gamma} \in \mathcal{W}_h$ satisfying

$$
\begin{aligned}
& a_h(\overline{\mathbf{u}}_{\varepsilon\gamma h}, \mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon\gamma h}) + b_h(\overline{\mathbf{u}}_{\varepsilon\gamma h}, \overline{\mathbf{u}}_{\varepsilon\gamma h}, \mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon\gamma h}) + \\
& + b_{sh}(T_h - T_{0h}, \mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon\gamma h}) + j_{\gamma h}(\mathbf{v}) - j_{\gamma h}(\overline{\mathbf{u}}_{\varepsilon\gamma h}) + j_{g\gamma h}(\mathbf{v}_h) - j_{g\gamma h}(\overline{\mathbf{u}}_{\varepsilon\gamma h}) + \\
& + \frac{1}{\varepsilon} c_h(\overline{\mathbf{u}}_{\varepsilon\gamma h}, \mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon\gamma h}) \geq S_h(\mathbf{v}_h - \overline{\mathbf{u}}_{\varepsilon\gamma h}) \quad \forall \mathbf{v}_h \in \mathcal{W}_h\,. \qquad (45)
\end{aligned}
$$

It is easy to show that the functionals $j_{\gamma h}(\mathbf{v})$ and $j_{g\gamma h}(\mathbf{v})$ are convex and continuous, and therefore, the problem (45) has a unique solution $\overline{\mathbf{u}}_{\varepsilon\gamma h} \in \mathcal{W}_h$. As a result we have the following result:

Theorem 4 *Let $\mathbf{u}_\varepsilon = \overline{\mathbf{u}}_\varepsilon + \mathbf{u}_1$, where $\overline{\mathbf{u}}_\varepsilon$ is a solution of the penalized problem with homogenous condition on Γ_u $\mathbf{u}_{\varepsilon h} = \overline{\mathbf{u}}_{\varepsilon h} + \mathbf{u}_1$, $\mathbf{u}_{\varepsilon\gamma h} = \overline{\mathbf{u}}_{\varepsilon\gamma h} + \mathbf{u}_1$ for all $\varepsilon, \gamma, h > 0$. Let $(\mathbf{u}, T)$ be the solution of the problem $(\mathcal{P}_{sf})_v$. Then*

$$
\begin{aligned}
&\text{(i)}\ \mathbf{u}_\varepsilon \to \mathbf{u} \text{ strongly in } H^{1,N}(\Omega) \text{ when } \varepsilon \to 0\,, \\
&\text{(ii)}\ \mathbf{u}_{\varepsilon h} \to \mathbf{u}_\varepsilon \text{ strongly in } H^{1,N}(\Omega) \text{ when } h \to 0\,, \\
&\text{(iii)}\ \mathbf{u}_{\varepsilon\gamma h} \to \mathbf{u}_{\varepsilon h} \text{ strongly in } H^{1,N}(\Omega) \text{ when } \gamma \to 0\,, \\
&\text{(iv) the semi} - \text{implicit scheme for thermal part of the problem} \\
&\quad \text{is stable and convergent.} \qquad (46)
\end{aligned}
$$

Then the visco-plastic part of the problem leads to solving the non-linear algebraic system, which can be solved by e.g. the Newton iterative method.

5 Conclusion

A thermal convection problem of Boussinesq fluid with infinite Prandtl number in a spherical shell has been used to describe the movements of the Earth's mantle. Since from the mechanical point of the view the Bingham model represents a generalization of the Newtonian viscous incompressible fluid, it has been used for mathematical models of Earth's mantle movements, where the global Earth's data can be used. We have presented the corresponding variational problem, which leads to the coupled system of variational equality and variational inequality. Numerical solution in its dynamical and stationary flow formulations are discussed.

Acknowledgements

The author would like to thank to Dr. Ivan Hlaváček, DrSc. as well as to both reviewers for a critical reading of the manuscript and their helpful comments.

References

[1] B. Awbi, L. Selmani, M. Sofonea, *Variational Analysis of a Frictional Contact Problem for the Bingham Fluid*; *Int. J. Appl. Math. and Comp. Sci* **9** (1999) 371-385.

[2] D. Bercovici, G. Schubert, G.A. Glatzmaier, A.I. Zebib, *Three-Dimensional Thermal Convection in a Spherical Shell*; *J. Fluid. Mech.* **206** (1989) 75-104.

[3] N. Cristescu, *Plastic Flow Through Conical Converging Dies, Using a Viscoplastic Constitutive Equations*; *International Journal of Mechanical Sciences* **17** (1975) 425-433.

[4] G. Duvaut, J.L. Lions, *Inequalities in Mechanics and Physics*; *Springer, Berlin* (1976).

[5] G.A. Glatzmaier, *Numerical Simulations of Mantle Convection: Time Dependent, Three-Dimensional, Compressible, Spherical Shell*; *Geophys. Astrophys. Fluid. Dyn.* **43** (1988) 223-264.

[6] R. Glowinski, J.L. Lions, R. Trémolières, *Numerical Analysis of Variational Inequalities*; *North-Holland, Amsterdam* (1976).

[7] I.R. Ionescu, M. Sofonea, *Functional and Numerical Methods in Viscoplasticity*; *Oxford Univ. Press, Oxford* (1993).

[8] J. Nedoma, *On the New Problem in the Potential Gravity Field*; in: *Král, J. Lukeš, J., Netuka, I., Veselý, J. (Eds), Potential Theory, Plenum Press, New York, London* (1988) 255-262.

[9] J. Nedoma, *Equations of Magnetodynamics in Incompressible Thermo-Bingham's Fluid Under the Gravity Effect*; *J.Comput. Appl. Math.* **59** (1995) 109-128.

[10] J. Nedoma, *Nonlinear Analysis of the Generalized Thermo-Magneto-Dynamic Problem*; *J.Comput. Appl. Math.* **63** (1995) 1-3 393-402.

[11] J. Nedoma, *On a Coupled Stefan-Like Problem in Thermo-Visco-Plastic Rheology*; *J.Comput. Appl. Math.* **84** (1997) 45-80.

[12] J. Nedoma, *Numerical Modelling in Applied Geodynamics*; *J.Wiley&Sons, Chichester* (1998).

[13] J. Nedoma, *Analysis of a Coupled System of Equations of a Global Dynamic Model of the Earth*; *Math.&Comput. in Simulation* **50** (1999) 265-283.

[14] J. Nedoma, *Numerical Solution of a Stefan-Like Problem in Bingham Rheology*; *Math.&Comput. in Simulation* **61** (2003) 271-281.

[15] W.R. Peltier, (Ed.), *Mantle Convection. Plate Tectonics and Global Dynamics*; *Gordon and Breach, New York* (1989).

[16] M. Tabata, *Finite Element Approximation to Infinite Prandtl Number Boussinesq Equations with Slip Boundary Condition*; *Proc. ECCOMAS'98, J. Wiley&Sons* (1998) 22-27.

[17] M. Tabata, A. Suzuki, *A Stabilized Finite Element Method for the Rayleigh-Bénard Equations with Infinite Prandtl Number in a Spherical Shell*; *Comput. Methods Appl. Mech. Engrg.* **190** (2000) 387-402.

[18] R. Temam, *Navier-Stokes Equations*; *Theory and Numerical Methods, North-Holland, Amsterdam* (1979).