Iterative solvers for stochastic Galerkin method

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Outline

Problems with parametric (uncertain) data

As an example,

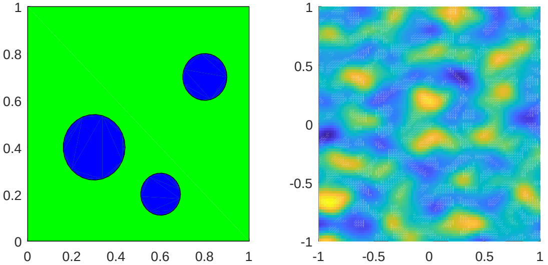
$$-\nabla \cdot a(\boldsymbol{x}, \boldsymbol{\xi}) \nabla u(\boldsymbol{x}, \boldsymbol{\xi}) = f(\boldsymbol{x}), \qquad (\boldsymbol{x}, \boldsymbol{\xi}) \in D \times \Gamma,$$

with $u(\boldsymbol{x}, \boldsymbol{\xi}) = 0$ on $\partial D \times \Gamma$, data $0 < \alpha_1 \leq a(\boldsymbol{x}, \boldsymbol{\xi}) \leq \alpha_2 < \infty$.

Parametric data/random field $a(\boldsymbol{x}, \boldsymbol{\xi})$

a) Left:
well separated domains
with different characteristics
$a(\boldsymbol{x}, \boldsymbol{\xi})$
$= a_0(\boldsymbol{x}) + \chi_1(\boldsymbol{x})\xi_1 + \chi_2(\boldsymbol{x})\xi_2$

b) Right:
Karhunen-Loéve expansion,
truncated
(covariance, eigenvectors)



Karhunen-Loéve expansion

Numerical computation needs discrete finite random field $a(\boldsymbol{x},\omega)$.
Covariance operator c,

$$C(g)(\boldsymbol{x}) = \int_D c(\boldsymbol{x},\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \qquad c(\boldsymbol{x},\boldsymbol{y}) = \mathrm{cov}(a(\boldsymbol{x},\omega),a(\boldsymbol{y},\omega))$$

with eigenvalues λ_k and eigenfunctions $a_k(\boldsymbol{x})$, then

$$a(\boldsymbol{x},\omega) = a_0(\boldsymbol{x}) + \sum_{k=1}^{\infty} \xi_k(\omega)\sqrt{\lambda_k}\,a_k(\boldsymbol{x}),$$

where ξ_k are uncorelated random variables with zero mean and unit variance.
Truncation, check $a_{\mathrm{trunc}}(\boldsymbol{x},\omega) > 0$.

Measure space $L^2_\rho(\Gamma)$

Doob-Dynkin lemma: measure space $L^2_\rho(\Gamma)$, $\rho(\xi) = \mathrm{d}P/\mathrm{d}\xi$ (instead of (Ω,Σ,P))

$$a(\boldsymbol{x},\omega) := a(\boldsymbol{x},\boldsymbol{\xi}(\omega)), \qquad \boldsymbol{\xi}(\omega) = (\xi_1(\omega),\dots,\xi_{N_\xi}(\omega)),$$

where the random variables $\xi_i(\omega)$ are iid with the joint probability density

$$\rho(\boldsymbol{\xi}) = \prod_{i=1}^{N_\xi} \rho_i(\xi_i) \quad \text{and} \quad \Gamma = \prod_{i=1}^{N_\xi} \Gamma_i = \prod_{i=1}^{N_\xi} \mathrm{Im}(\xi_i).$$

Data $a(\boldsymbol{x},\boldsymbol{\xi})$

a) linear w.r.t. stochastic part

$$a(\boldsymbol{x},\boldsymbol{\xi}) = a_0(\boldsymbol{x}) + \sum_{i=1}^{N_\xi} a_i(\boldsymbol{x})\xi_i, \qquad \text{or} \qquad a(\boldsymbol{x},\boldsymbol{\xi}) = a_0(\boldsymbol{x}) + \sum_{i=1}^{N_\xi} \chi_i(\boldsymbol{x})\xi_i$$

b) non-linear w.r.t. stochastic part

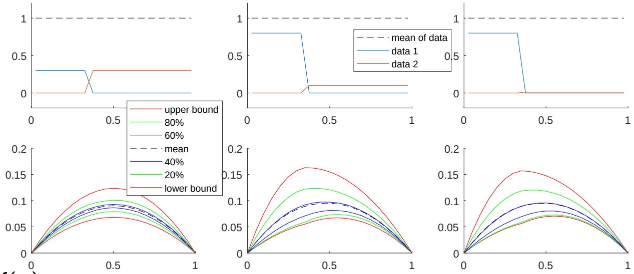
$$a(\boldsymbol{x},\boldsymbol{\xi}) = \exp\left(\widetilde{a}_0(\boldsymbol{x}) + \sum_{i=1}^{N_\xi} \widetilde{a}_i(\boldsymbol{x})\xi_i\right) = \sum_{j=0}^{N_a} a_j(\boldsymbol{x})p_j(\boldsymbol{\xi}),$$

Stochastic variables / parameters $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_{N_\xi}) \in \Gamma$

Probability distribution $N(0,1)$, $U(-1,1)$, etc.
Probability density / weight function ρ

Where we can meet parametric problems

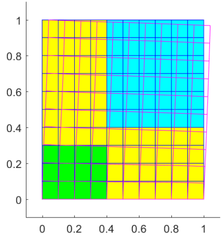
a) Studying dependency of u on the data

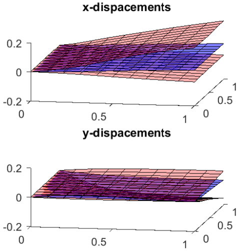


iso-lines

$-(a(x,\boldsymbol{\xi})u(x,\boldsymbol{\xi})')' = f(x)$

$\boldsymbol{\xi} = ({\color{blue}\xi_1}, {\color{red}\xi_2})$





iso-surfaces

linear elasticity

stiffness in three domains ${\color{green}2} : {\color{cyan}3} : {\color{orange}1}$

$\boldsymbol{\xi} = ({\color{green}\xi_1}, {\color{cyan}\xi_2}, {\color{orange}\xi_3})$

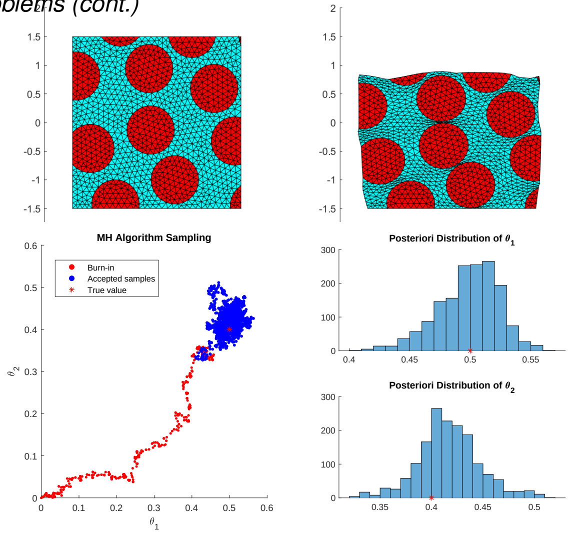
Where we meet parametric problems (cont.)

b) Identifying parameters, inverse problems

linear elasticity
$\boldsymbol{\xi} = (\xi_1, \xi_2)$

Bayes methods
Metropolis-Hastings method
surrogate models

(figures by L.Gaynutdinova)



Blaheta, Béreš, Domesová, Pan, A comparison of deterministic and Bayesian inverse with application in micromechanics, Applications of Mathematics, 2018

Solution methods for parametric problems

a) Monte Carlo methods

+ nonintrusive, multilevel Monte Carlo methods, universal
- time consuming (unless applied in parallel), generating samples, lack of guaranteed error bounds

b) Collocation methods (w.r.t. stochastic variables)

+ nonintrusive, sparse grids, nested grids (Clenshaw-Curtis quadrature)
- curse of dimensionality, discrete approximation measure, lack of guaranteed error bounds

c) Stochastic Galerkin methods / stochastic finite element methods

+ integral approximation measure, various post-processing of results, guaranteed error bounds
- intrusive (unless using double-orthogonal approximation polynomials), large linear systems, coupled problem, curse of dimensionality

Stochastic Galerkin method / stochastic FE method

As example

$$-\nabla \cdot a(\boldsymbol{x},\boldsymbol{\xi})\nabla u(\boldsymbol{x},\boldsymbol{\xi}) = f(\boldsymbol{x}), \qquad (\boldsymbol{x},\boldsymbol{\xi}) \in D \times \Gamma,$$

with $u(\boldsymbol{x},\boldsymbol{\xi}) = 0$ on $\partial D \times \Gamma$.

Stochastic variational form. Find $u \in V = H_0^1(D) \times L_\rho^2(\Gamma) = L_\rho^2(\Gamma, H_0^1(D))$ (Bochner space) such that

$$\mathscr{A}(u,v) = \mathscr{F}(v), \quad v \in V.$$

Equality of moments.
Energy inner product and linear functional

$$\begin{aligned}\mathscr{A}(u,v) &= \int_\Gamma \int_D \nabla v(\boldsymbol{x},\boldsymbol{\xi}) \cdot a(\boldsymbol{x},\boldsymbol{\xi})\nabla u(\boldsymbol{x},\boldsymbol{\xi})\rho(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\xi}\\ \mathscr{F}(v) &= \int_\Gamma \int_D f(\boldsymbol{x})v(\boldsymbol{x},\boldsymbol{\xi})\rho(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\xi}.\end{aligned}$$

Regularity and a priori convergence estimates
Babuska, Nobile, Tempone, Ghanem, Zouraris; Gittelson, 2009; Bespalov, Powell, Silvester, 2012

Discretization

Solution $u(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{r,j=1}^{N_{\mathrm{FE}},N_{\mathrm{pol}}} \mathrm{u}_{(j-1)N_{\mathrm{FE}}+r}\,\phi_r(\boldsymbol{x})\Psi_j(\boldsymbol{\xi})$

Approximation basis functions $\phi_r(\boldsymbol{x})\Psi_j(\boldsymbol{\xi}) \in V^{\mathrm{FE}} \times V^{\mathrm{pol}} \subset V$

Finite element basis functions $\phi_r(\boldsymbol{x}) \in V^{\mathrm{FE}} \subset H_0^1(D)$

Orthogonal w.r.t. weight ρ polynomials $\Psi_j(\boldsymbol{\xi}) = \psi_{j1}(\xi_1)\cdots\psi_{jN_\xi}(\xi_{N_\xi}) \in V^{\mathrm{pol}} \subset L_\rho^2(\Gamma)$

Polynomials

Hermite pol. $\rho(\xi) = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-\xi^2/2}$, Legendre pol. $\rho(\xi) = \frac{1}{2}\chi_{\langle -1,1\rangle}$, etc.

Complete polynomials (total degree $\le d$) $N_{\mathrm{pol}} = \binom{N_\xi + d}{N_\xi}$

Tensor product (degrees $\le d_i$ at ξ_i) $N_{\mathrm{pol}} = \prod_{i=1}^{N_\xi}(d_i+1)$

Matrices

$$\begin{aligned} \mathsf{A}_{(k-1)N_{\mathsf{FE}}+s,(j-1)N_{\mathsf{FE}}+r} &= \mathscr{A}(\psi_s\Phi_k, \psi_r\Phi_j) \\ &= \int_\Gamma \int_D \nabla\psi_s\Phi_k \cdot a(\boldsymbol{x},\boldsymbol{\xi})\nabla\psi_r\Phi_j\,\rho\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\xi} \\ &= \int_\Gamma \Phi_k\Phi_j\rho \int_D \nabla\psi_s \cdot a(\boldsymbol{x},\boldsymbol{\xi})\nabla\psi_r\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\xi} \end{aligned}$$

depend on data $a(\boldsymbol{x},\boldsymbol{\xi})$; e.g. (slide 5)

$$\begin{aligned} \mathsf{A}_{(k-1)N_{\mathsf{FE}}+s,(j-1)N_{\mathsf{FE}}+r} &= \int_\Gamma \Phi_k\Phi_j\rho \int_D \sum_{l=0}^{N_a} a_l(\boldsymbol{x})p_l(\boldsymbol{\xi})\nabla\psi_s\cdot\nabla\psi_r\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\xi} \\ &= \sum_{l=0}^{N_a}\int_\Gamma p_l(\boldsymbol{\xi})\Phi_k\Phi_j\rho\,\mathrm{d}\boldsymbol{\xi}\int_D a_l(\boldsymbol{x})\nabla\psi_s\cdot\nabla\psi_r\,\mathrm{d}\boldsymbol{x} \\ &= \sum_{l=0}^{N_a}(\mathsf{G}_l)_{kj}\cdot(\mathsf{K}_l)_{sr} \end{aligned}$$

A is s.p.d. (unless, e.g., Gauss distribution and high degree approximation)

$$\mathsf{b}_{(k-1)N_{\mathsf{FE}}+s} = \mathscr{F}(\psi_s\Phi_k) = \int_\Gamma\int_D f\psi_s\Phi_k\,\rho\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\xi}$$

Ernst, Ullmann, 2008

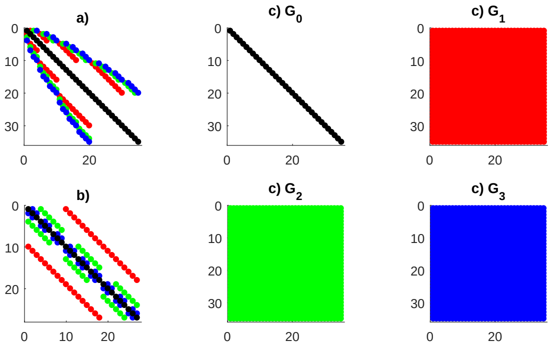
Structures of matrices

Matrix A is not built. Sum of tensor products $A = \sum_{l=0}^{N_\xi} G_l \otimes K_l$

Examples:
a) $a(\boldsymbol{x}, \boldsymbol{\xi}) = a_0(\boldsymbol{x}) + \sum_{i=1}^{3} a_i(\boldsymbol{x})\xi_i$, complete polynomials, $d = 4$
b) $a(\boldsymbol{x}, \boldsymbol{\xi}) = a_0(\boldsymbol{x}) + \sum_{i=1}^{3} a_i(\boldsymbol{x})\xi_i$, tensor product polynomials, $d_1 = d_2 = d_3 = 2$
c) $a(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{l=0}^{3} a_l(\boldsymbol{x}) p_l(\boldsymbol{\xi})$, $N_a = 3$, complete polynomials, $d = 4$

G_0, G_1, G_2, G_3

Every dot is a multiple of an $N_{FE} \times N_{FE}$ "stiffness matrix" K_l



Structure of matrices

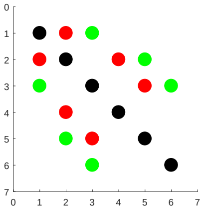
Linear $a(\boldsymbol{x},\xi) = a_0(\boldsymbol{x}) + \xi_1 a_1(\boldsymbol{x}) + \xi_2 a_2(\boldsymbol{x})$:

$N_\xi = 2$, uniform distribution of ξ_i, Legendre polynomials, complete pol. $d = 2$

$$
\mathrm{A} = \left(\begin{array}{c|cc|ccc}
\mathrm{K}_0 & \frac{1}{\sqrt{3}}\mathrm{K}_1 & \frac{1}{\sqrt{3}}\mathrm{K}_2 & 0 & 0 & 0 \\
\hline
\frac{1}{\sqrt{3}}\mathrm{K}_1 & \mathrm{K}_0 & 0 & \frac{2}{\sqrt{15}}\mathrm{K}_1 & \frac{1}{\sqrt{3}}\mathrm{K}_2 & 0 \\
\frac{1}{\sqrt{3}}\mathrm{K}_2 & 0 & \mathrm{K}_0 & 0 & \frac{1}{\sqrt{3}}\mathrm{K}_1 & \frac{2}{\sqrt{15}}\mathrm{K}_2 \\
\hline
0 & \frac{2}{\sqrt{15}}\mathrm{K}_1 & 0 & \mathrm{K}_0 & 0 & 0 \\
0 & \frac{1}{\sqrt{3}}\mathrm{K}_2 & \frac{1}{\sqrt{3}}\mathrm{K}_1 & 0 & \mathrm{K}_0 & 0 \\
0 & 0 & \frac{2}{\sqrt{15}}\mathrm{K}_2 & 0 & 0 & \mathrm{K}_0
\end{array}\right),
$$

K_0, K_1, and K_2 are "stiffness matrices"
corresponding to $a_0(\boldsymbol{x})$, $a_1(\boldsymbol{x})$, and $a_2(\boldsymbol{x})$, respectively

stiffness matrices
$(\mathrm{K}_i)_{rs} = \int_D a_i(\boldsymbol{x})\nabla\psi_r(\boldsymbol{x})\nabla\psi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$
Jacobi matrices
$(\mathrm{G}_i)_{jk} = \int_\Gamma \xi_i \Phi_j(\boldsymbol{\xi})\Phi_k(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$
$(\mathrm{G}_0)_{jk} = \int_\Gamma \Phi_j(\boldsymbol{\xi})\Phi_k(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \delta_{jk}$



Double orthogonal polynomials

$p_0, p_1, p_2, \ldots$ (infinite set) orthogonal on $\Gamma \subset \mathbb{R}$

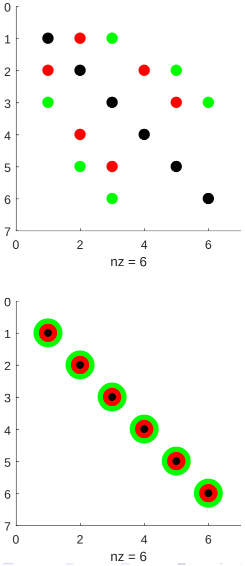
$$\int_\Gamma p_j(z) p_k(z) \rho(z) \mathrm{d}z = \delta_{jk}$$

$\widetilde{p}_1, \widetilde{p}_2, \ldots \widetilde{p}_m$ (finite set) double-orthogonal on $\Gamma \subset \mathbb{R}$, Lagrange polynomials with the set of nodes - roots of p_m, all of the degree $m-1$

$$\int_\Gamma \widetilde{p}_j(z) \widetilde{p}_k(z) \rho(z) \mathrm{d}z = \delta_{jk}$$

$$\int_\Gamma z\, \widetilde{p}_j(z) \widetilde{p}_k(z) \rho(z) \mathrm{d}z = \delta_{jk}$$

A is block diagonal matrix with different blocks.



Solution methods

Conjugate gradient method with preconditioning

Preconditioning of $\mathrm{Au} = \mathrm{b}$ is getting M such that $\mathrm{M}^{-1}\mathrm{Au} = \mathrm{M}^{-1}\mathrm{b}$ is better solvable than $\mathrm{Au} = \mathrm{b}$, and $\mathrm{Mv} = \mathrm{c}$ is easy to solve. Also in an abstract form.

Preconditioning

1) Multigrid w.r.t. physical variable
Brezina, et al., 2014, Elman, Furnival, 2007

2) Multilevel w.r.t. stochastic variable ... focused in this talk

3) Reduced basis, low rank approximations, rational Krylov subspace, etc.

vector $u \to$ tensorised matrix U, and

$$\mathrm{b} = \mathrm{Au} = \left(\sum_{i=0}^{N_a} \mathrm{G}_i \otimes \mathrm{K}_i\right) u \qquad \text{is the same as} \qquad \mathrm{B} = \sum_{i=0}^{N_a} \mathrm{K}_i \mathrm{U} \mathrm{G}_i^T$$

Matthies et al., 2014; Powell, Silvester, Simoncini, 2016; Powell, Silvester, Simoncini, 2018; Audouze, Nair, 2019

Multilevel preconditioning with respect to stochastic variable $\left(A = \sum_{i=0}^{N_a} G_i \otimes K_i\right)$

Mean based - diagonal blocks - Powell, Elman, 2009; etc.

$$M^{\text{mean}} = G_0 \otimes K_0$$

Kronecker product preconditioner - Ullmann, 2010

$$M^{\text{Kron}} = G_0 \otimes K_0 + \sum_{i=1}^{N_a} \beta_i G_i \otimes K_0, \qquad \beta_i = \frac{\operatorname{tr}({K_0}^T K_i)}{\operatorname{tr}({K_0}^T K_0)}$$

Symmetric block Gauss-Seidel - Bespalov, Loghin, Youngnoi, arXiv 2020

$$M^{\text{SBGS}} = \left(G_0 \otimes K_0 + \sum_{i=1}^{N_a} L_i \otimes K_i\right)(G_0 \otimes K_0)^{-1}\left(G_0 \otimes K_0 + \sum_{i=1}^{N_a} {L_i}^T \otimes K_i\right), \quad L_i + {L_i}^T = G_i$$

Two-by-two blocks and Schur complement - Sousedík, Ghanem, Phips, 2013

Block diagonal precoditioner with large blocks - P., Kubínová, 2020

Overview - Crowder, Adaptive and Multilevel Stochastic Galerkin Finite Element Methods, Ph.D. Thesis, 2020

Numerical experiments

1D diffusion equation, $N_{\mathrm{FE}} = 20$, complete polynomials, maximum degree d, number of stoch. variables N_ξ, ξ_i uniformly distributed in $[-1, 1]$

Table: Mean based, Kronecker and SBGS preconditioning.

		$\kappa(M^{-1}A)$				CG steps			
N_ξ	d	no	mean	Kron	SBGS	no	mean	Kron	SBGS
1	1	252.8	2.1	1.6	1.1	39	10	8	5
1	3	317.5	3.1	2.0	1.2	71	13	9	5
1	7	345.4	3.7	2.3	1.2	103	14	10	5
2	1	256.2	2.1	1.6	1.1	56	10	8	5
2	3	335.5	3.9	2.5	1.3	99	14	11	6
2	7	387.9	6.1	3.6	1.5	125	17	13	6
3	1	262.6	2.1	1.6	1.2	60	10	7	5
3	3	372.6	4.2	2.8	1.4	112	14	11	6
3	7	462.6	7.8	5.0	1.8	142	20	15	7

Our contribution

Improving guaranteed spectral bounds for preconditioned matrix $\mathsf{M}^{-1}\mathsf{A}$ based on

— orthogonal polynomial properties
— data of problems associated to A and M - element-by-element
— for many kinds of distribution of data

P., 2016; Kubínová, P., 2020, Plešinger, P., 2018

Our approach - connected to and based on classical condition number estimates for algebraic multi-level (AML) preconditioning

Eijkhout, Vassilevski, Axelsson, Neytcheva, Blaheta, Kraus

Patterns of blocks of M:

$$\mathsf{M}^{\mathsf{C}} = \left(\begin{array}{c|cc|ccc} X & & & & & \\ \hline & X & & & & \\ & & X & & & \\ \hline & & & X & & \\ & & & & X & \\ & & & & & X \end{array}\right), \qquad \mathsf{M}^{\mathsf{C}} = \left(\begin{array}{c|cc|ccc} X & X & X & & & \\ \hline X & X & & & & \\ X & & X & & & \\ \hline & & & X & & \\ & & & & X & \\ & & & & & X \end{array}\right)$$

Our contribution (cont.)

Patterns of blocks of M:

$$\mathrm{M}^{\mathrm{TP}} = \left(\begin{array}{ccc|ccc|ccc} X & & & & & & & & \\ & X & & & & & & & \\ & & X & & & & & & \\ \hline & & & X & & & & & \\ & & & & X & & & & \\ & & & & & X & & & \\ \hline & & & & & & X & & \\ & & & & & & & X & \\ & & & & & & & & X \end{array}\right), \quad \mathrm{M}^{\mathrm{C}} = \left(\begin{array}{c|cc|ccc} X & & & & & \\ \hline & X & & & & \\ & & X & & & \\ \hline & & & X & & \\ & & & & X & \\ & & & & & X \end{array}\right),$$

$$\mathrm{M}^{\mathrm{TP}} = \left(\begin{array}{ccc|ccc|ccc} X & X & & & & & & & \\ X & X & X & & & & & & \\ & X & X & & & & & & \\ \hline & & & X & X & & & & \\ & & & X & X & X & & & \\ & & & & X & X & & & \\ \hline & & & & & & X & X & \\ & & & & & & X & X & X \\ & & & & & & & X & X \end{array}\right).$$

$$\mathrm{M}^{\mathrm{TP}} = \left(\begin{array}{ccc|ccc|ccc} X & X & & X & & & & & \\ X & X & X & & X & & & & \\ & X & X & & & X & & & \\ \hline X & & & X & X & & & & \\ & X & & X & X & X & & & \\ & & X & & X & X & & & \\ \hline & & & & & & X & X & \\ & & & & & & X & X & X \\ & & & & & & & X & X \end{array}\right), \quad \mathrm{M}^{\mathrm{C}} = \left(\begin{array}{c|cc|ccc} X & X & X & & & \\ \hline X & X & & & & \\ X & & X & & & \\ \hline & & & X & & \\ & & & & X & \\ & & & & & X \end{array}\right).$$

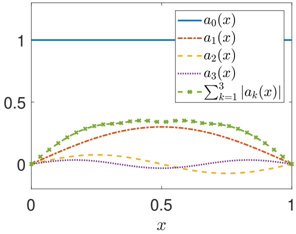
Our contribution (cont.)

Numerical experiment.

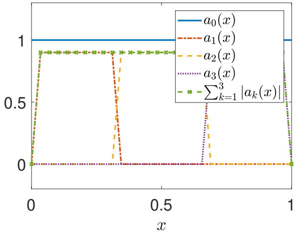
$$-(a(x,\boldsymbol{\xi})u(x,\boldsymbol{\xi})')' = f(x),$$

$D = (0,1)$, $N_\xi = 3$, $a(x,\boldsymbol{\xi}) = a_0(x) + \xi_1 a_1(x) + \xi_2 a_2(x) + \xi_3 a_3(x)$, $\xi_i \in [-1,1]$, $i = 1,2,3$.

Problem 1:



Problem 2:



Our contribution (cont.)

Table: Block-diagonal preconditioning of Problem 1 and Problem 2. New and classical bounds.

d	$\kappa(A)$	$\underline{c}_{class}$	$\underline{c}$	$\lambda_{min}(M^{-1}A)$	$\lambda_{max}(M^{-1}A)$	$\overline{c}$	$\overline{c}_{class}$	$\overline{c}/\underline{c}$
Problem 1								
1	458.42	0.76	0.80	0.83	1.17	1.20	1.24	1.51
2	498.47	0.68	0.73	0.76	1.24	1.27	1.32	1.75
...								
6	546.55	0.61	0.67	0.69	1.31	1.33	1.39	2.26
7	550.80	0.61	0.66	0.68	1.32	1.34	1.39	2.29
Problem 2								
1	947.79	-0.65	0.45	0.45	1.56	1.56	2.65	3.43
2	1596.34	-1.21	0.26	0.26	1.74	1.74	3.21	6.57
...								
6	4576.93	-1.71	0.10	0.10	1.90	1.90	3.71	19.34
7	5294.63	-1.74	0.09	0.09	1.91	1.91	3.74	21.80

P., 2015, 2016, 2017; Kubínová, P., 2020

Conclusion

- PDE with (stochastic) parameters - many methods, demanding
- Variational approach - stochastic Galerkin method
  Precoditioning – a posteriori error estimates – adaptivity
  A posteriori error estimates
  Bespalov, Powell, Silvester, 2014; Eigel, Merdon, 2014; Khan, Bespalov, Powell, Silvester, 2020
  Adaptivity
  Eigel, Gittelson, Schwab, Zander, 2014; P., 2015; Bespalov, Praetorius, Ruggeri, 2020
- Relatively short history; many recent results; still developing
  Using as much information about A and M as possible
- Spectral estimates of $M^{-1}A$ (matrix pencil) - sharp new bounds based on element-by-element approach
  P., 2016; Kubínová, P., 2017; Plešinger, P., 2018
- Estimates of all eigenvalues of preconditioned matrices
  Gergelits, Mardal, Nielsen and Strakoš, 2019; Gergelits, Nielsen, Strakoš, 2020;
  Ladecký, P., Zeman, 2020; P., Ladecký, submitted
  Our new method described in Martin Ladecký's talk at SNA 2021.

Thanks for your attention.

Thank you organizers, Stanislav, Dagmar, Hana, Jiří and Radim!